Numerics of Partial Differential Equations

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Mathematical Description of Fluid Dynamics

- Classification of partial differential equations
- Classification of the Euler and Navier-Stokes equations
- Flow phenomena: Shocks, rarefaction, contact discontinuity
- Numerical schemes

Classification of Partial Differential Equations

- Second-order Partial Differential Equations (PDE)
  - Elliptic, parabolic, hyperbolic
- Conservation equations
- Transport equations

Second-Order PDEs

$$\boxed{au_{xx} + bu_{xy} + cu_{yy} = h(u, u_x, u_y, x, y)}$$

notation: $u_{xx} = \frac{\partial^2 u}{\partial x^2}, \; u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, u_{yy} = \frac{\partial^2 u}{\partial y^2}$

$a, b, c \in \Re$	constant coefficients
$a = a(x, y), b = b(x, y), c = c(x, y)$	linear
$a = a(u), \; b = b(u), \; c = c(u)$	quasilinear
$a = a(u, u_x, u_y, x, y)$, b, c analog	non-linear

Classification of Second-Order PDEs

$$au_{xx} + bu_{xy} + cu_{yy} = h(u, u_x, u_y, x, y)$$

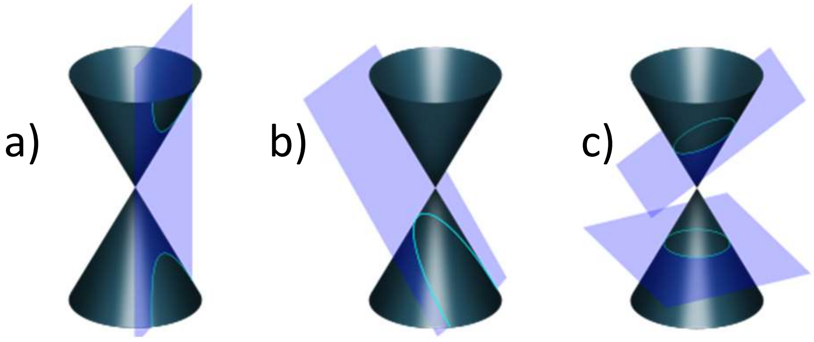
Criteria:

a) hyperbolic : $b^2 - 4ac > 0$

b) parabolic : $b^2 - 4ac = 0$

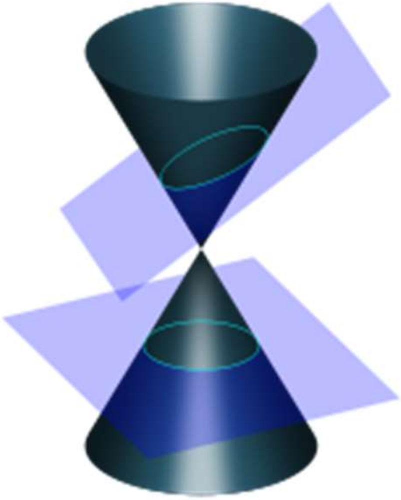
c) elliptic : $b^2 - 4ac < 0$

Analog: conic section $ax^2 + bxy + cy^2 = r$



Elliptical Differential Equations

- Physics:
  - Stationary problems: heat transport, potential flow, electric potential
  - Stationary subsonic flows
  - Membrane deflection
- Mathematical model:
  - Laplace equation $u_{xx} + u_{yy} = 0$
  - Poisson equation $u_{xx} + u_{yy} = f$
  - Helmholtz equation $u_{xx} + u_{yy} + ku = 0$



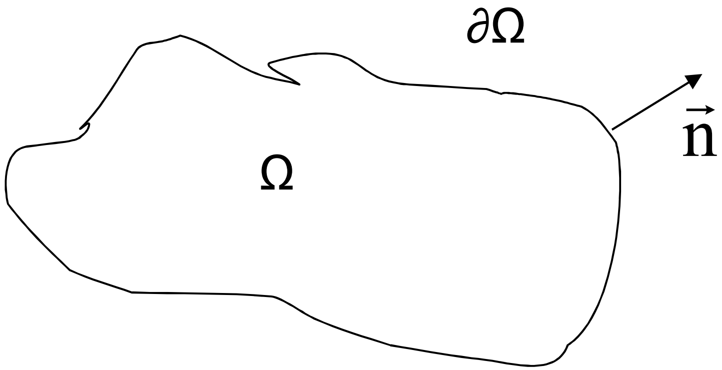
Constraints on Elliptical Differential Equations

- Boundary value problem

Dirichlet condition $u = f$

Neumann condition $\frac{\partial u}{\partial n} = g$

Robin condition $u + \alpha \nabla u \cdot n = h$



Parabolic Differential Equations

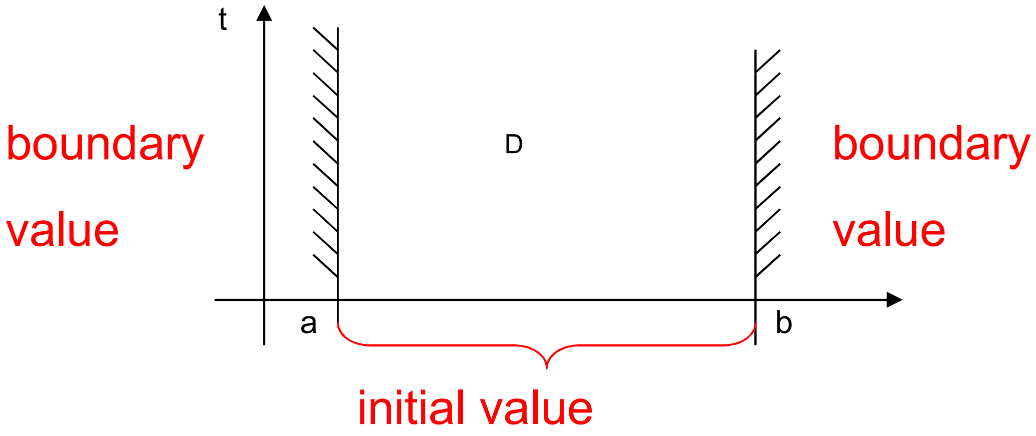
- Physics:
  - Heat transport, diffusion processes
  - Friction-viscosity
  - Transient processes

- Mathematical model:
  - Heat equation $u_t = \kappa \Delta u$
  - Heat equation 1D $u_t = \kappa u_{xx}$
- Limit for stationary processes (u_t=0):

  Parabolic → elliptic

Constraints on Parabolic Differential Equations

- Initial value and boundary value problem



$$u_t = \kappa u_{xx} \quad \text{in } [a, b] \times [0, T]$$

$$u(x,0) = g(x) \qquad \text{für } x \in [a, b]$$

$$u(a, t) = u_a(t) \qquad \text{für } t \in [0, T]$$

$$u(b, t) = u_b(t) \qquad \text{für } t \in [0, T]$$

Hyperbolic Differential Equation

- Physics:
  - Wave propagation, acoustic
  - Electromagnetic waves
  - Gas dynamics

- Mathematical model:
  - Wave equation $\mathrm{u}_{\mathrm{tt}} = c^2 \Delta u$
  - Wave equation 1D $\mathrm{u}_{\mathrm{tt}} = c^2 u_{xx}$

$$\text{transport velocity}: \mathrm{c} \neq 0$$

Constraints on Hyperbolic Differential Equations

- Initial value problem

$$u(x,0) = g(x) \quad \forall\, x \in \Re$$

Annotation: In practical simulations, the flow depends on initial values and boundary values (e.g. flow around an airfoil). The boundary values are not fixed, but depend on the solution of the flow. Boundary values have to satisfy compatibility conditions.

Hyperbolic Differential Equations of 1. Order

Wave equation 1D: $u_{tt} - c^2 \cdot u_{xx} = 0$

Transformation in system of first order:

New variables: $p = c \cdot u_x$, $q = u_t$

Resulting in two equations of first order

$$\begin{aligned} q_t - c \cdot p_x &= 0 \\ p_t - c \cdot q_x &= 0 \end{aligned}$$

system of first order

Hyperbolic Differential Equations of 1. Order

- Simplest hyperbolic equation in 1D:

$$u_t + au_x = 0$$ Linear transport equation

- Simplest hyperbolic equation in 2D:

$$u_t + au_x + bu_y = 0$$

- Simplest quasi-linear hyperbolic equation

$$u_t + uu_x = 0$$ Burgers' equation

Conservation Law in 1D

- Burgers' equation $u_t + uu_x = 0$ quasi-linear

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$ conservation form

- In general: $u_t + f(u)_x = 0$
  u,f,g are scalars or vectors.

$$u_t + Au_x = 0$$ quasi-linear

$$A = \frac{\partial f(u)}{\partial(u)}$$ Jacobian matrix

Conservation Law in 2D

- In general $u_t + f(u)_x + g(u)_y = 0$ conservation form

  u,f,g are scalars or vectors

$$u_t + Au_x + Bu_y = 0$$ quasi-linear

$$A = \frac{\partial f(u)}{\partial (u)}, \qquad B = \frac{\partial g(u)}{\partial (u)}$$ Jacobian matrix

Compressible Navier-Stokes Equation

$$\rho_t + \nabla \cdot (\rho v) = 0$$ conservation of mass

$$(\rho v)_t + \nabla \cdot ((\rho v) \circ v) + \nabla p = \nabla \cdot \tau + f$$ conservation of momentum

$$e_t + \nabla \cdot (v(e + p)) = \nabla(\tau \cdot v) - \nabla \cdot q + f \cdot v + Q$$ conservation of energy

ρ	density	p	pressure
v	velocity	τ	viscosity
e	energy	q	heat flux

equation of state $p = (\gamma - 1)\rho\varepsilon$, $e = \rho\varepsilon + \frac{1}{2}\rho(v \cdot v) + \rho\mathrm{gh}$

ideal gas

inner kinetic potential
energy

system of conservation laws/ equations

Euler Equations

$$\rho_t + \nabla \cdot (\rho v) = 0$$ conservation of mass

$$(\rho v)_t + \nabla \cdot ((\rho v) \circ v) + \nabla p = 0$$ conservation of momentum

$$e_t + \nabla \cdot (v(e + p)) = 0$$ conservation of energy

ρ	density	p	Pressure
v	velocity		
e	energy		

equation of state $p = (\gamma - 1)\rho\varepsilon \ ,\qquad e = \rho\varepsilon + \tfrac{1}{2}\rho(v \cdot v)$

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system of conservation laws/ equations

Theory of Characteristics

- Has to satisfy linear transport equation

$$u_t + a u_x = 0$$

$$C = \left\{ x = x(\mathrm{t}) \text{ with } \frac{\mathrm{d}\,x(t)}{\mathrm{d}\,t} = a \right\} \Rightarrow \quad \frac{\mathrm{d}\,u(x(t),t)}{\mathrm{d}\,t} = u_{\mathrm{t}} + u_{\mathrm{x}} \frac{\mathrm{d}\,x}{\mathrm{d}\,t} = u_{\mathrm{t}} + a u_{\mathrm{x}}$$

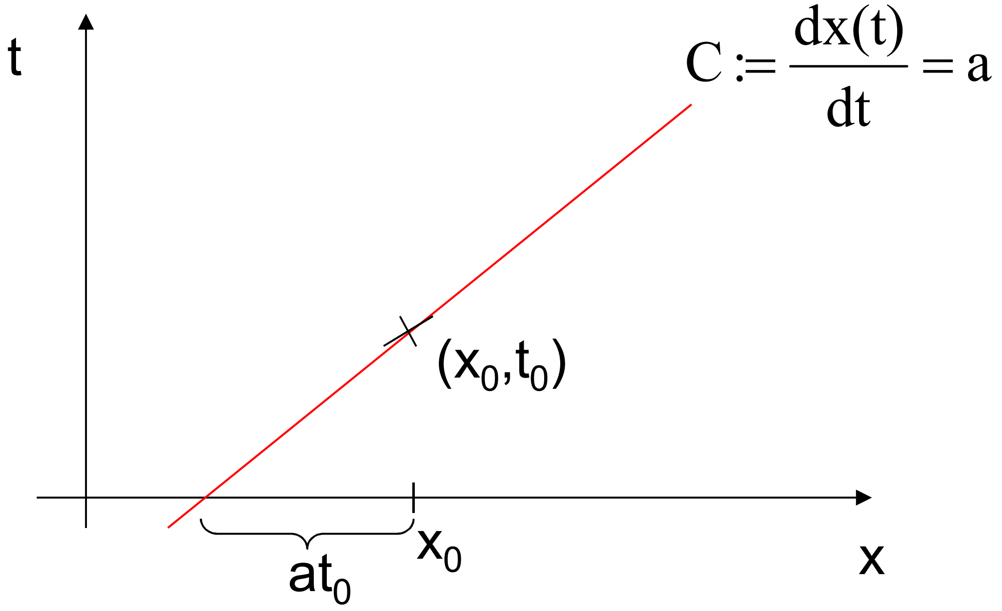
- Comparison of coefficients

$$\Rightarrow \quad \mathrm{C} : \frac{\mathrm{dx}}{\mathrm{dt}} = \mathrm{a(x,t)}$$

a = const. ➔ C is linear

u = const. on C ➔ u(x, t) = u(x - at, 0)

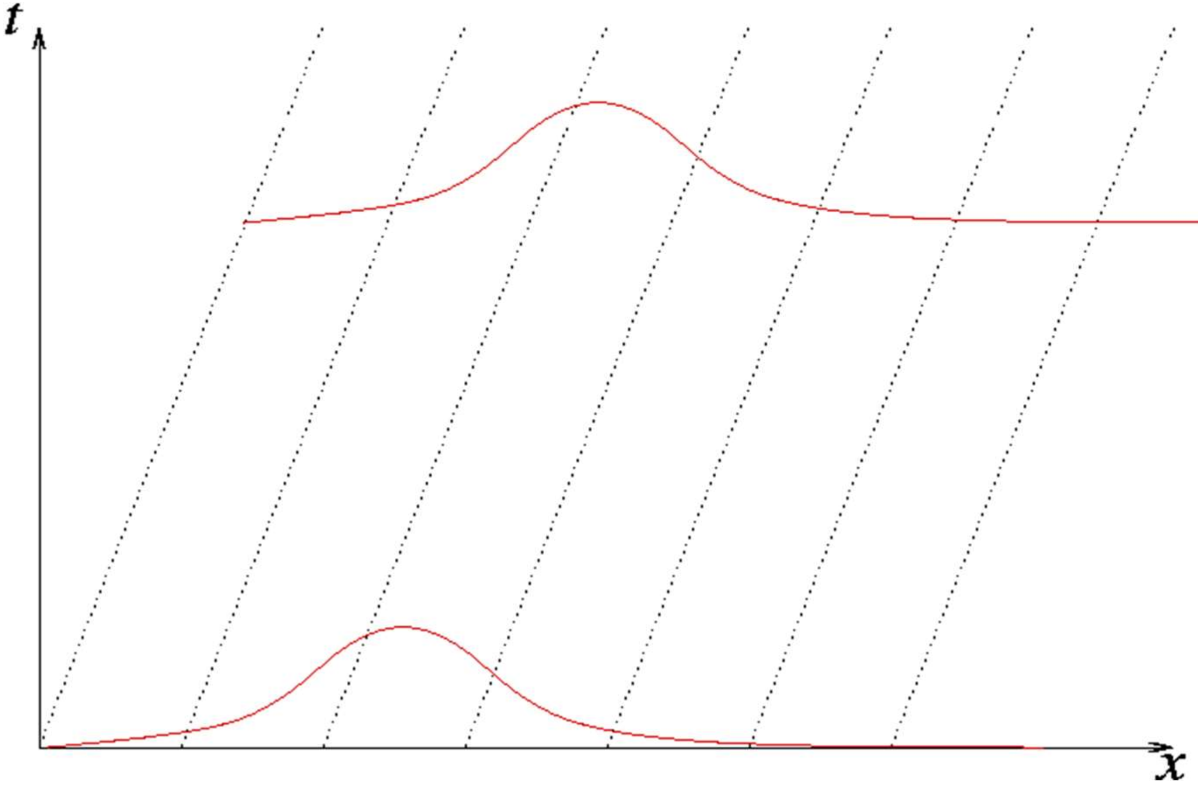
Solution of Initial Value Problem for Linear Transport Euqations



$u(x, t) = u_0(x - at)$ for every x, t

Information is transported along characteristics.

Solution is Constant along a Characteristic

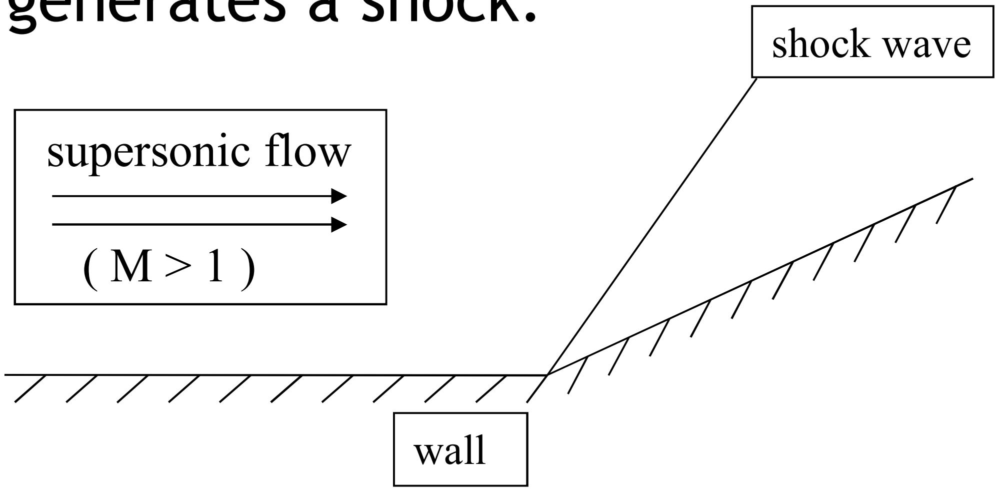


$x = a\,t + x_0$

Euler Equations – Flow Phenomena

The non-linear Euler equations allow for discontinuities in the solution, known as shocks. Shocks can be generated out of smooth initial values.

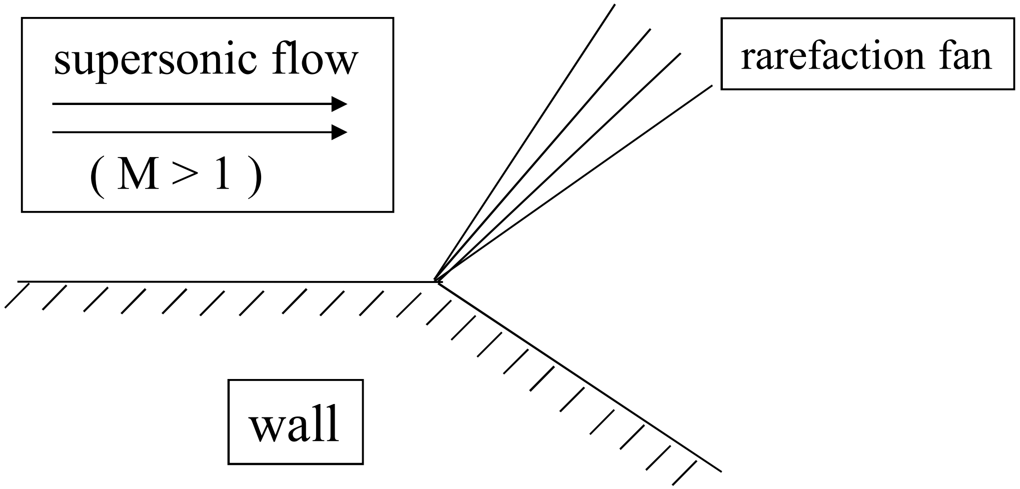
Example: The compression of a supersonic flows in front of a wall generates a shock.



Euler Equations – Flow Phenomena II

The non-linear Euler equations allow for smooth solutions, too. These phenomena are known as rarefaction fans.

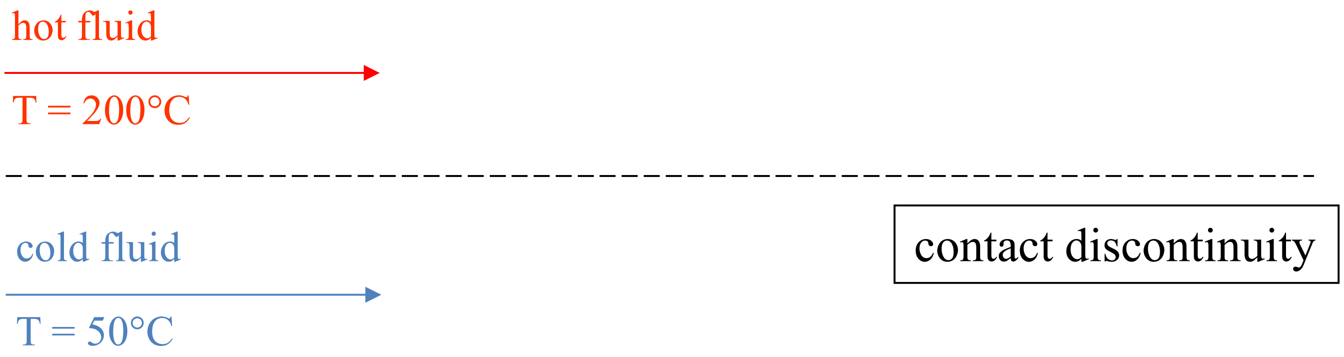
Example: Rarefaction of supersonic flows close to walls.



Euler Equations - Flow Phenomena III

The non-linear Euler equations allow for material interfaces (e.g. water-air, hot and cold fluid,..). These phenomena are called contact discontinuities.

Example: Separation of hot and cold fluids.



hot fluid

T = 200°C

cold fluid

T = 50°C

contact discontinuity

Properties of the Euler Equation

- The Euler equations are hyperbolic equations – non-linear wave equations.
- Problems: Shocks and Rarefaction

  Characteristics of non-linear equations depend on the solution and can intersect: The solution cannot be differentiated → The differential system of equations is not longer valid.
- The numerical method must handle all phenomena: Shock capturing

Compressible Navier-Stokes Equation - Type

$$\rho_t + \nabla \cdot (\rho v) = 0$$

conservation of mass

$$(\rho v)_t + \nabla \cdot ((\rho v) \circ v) + \nabla p = \nabla \cdot \tau + f$$

conservation of momentum

$$e_t + \nabla \cdot (v(e+p)) = \nabla(\tau \cdot v) - \nabla \cdot q + f \cdot v + Q$$

conservation of energy

ρ	density	p	pressure
v	velocity	τ	viscosity
e	energy	q	heat flux

Parabolic – Hyperbolic!!!

equation of state $p = (\gamma - 1)\rho\varepsilon$, $e = \rho\varepsilon + \frac{1}{2}\rho(v \cdot v) + \rho gh$

ideal gas

inner kinetic potential
energy

system of conservation laws/ equations

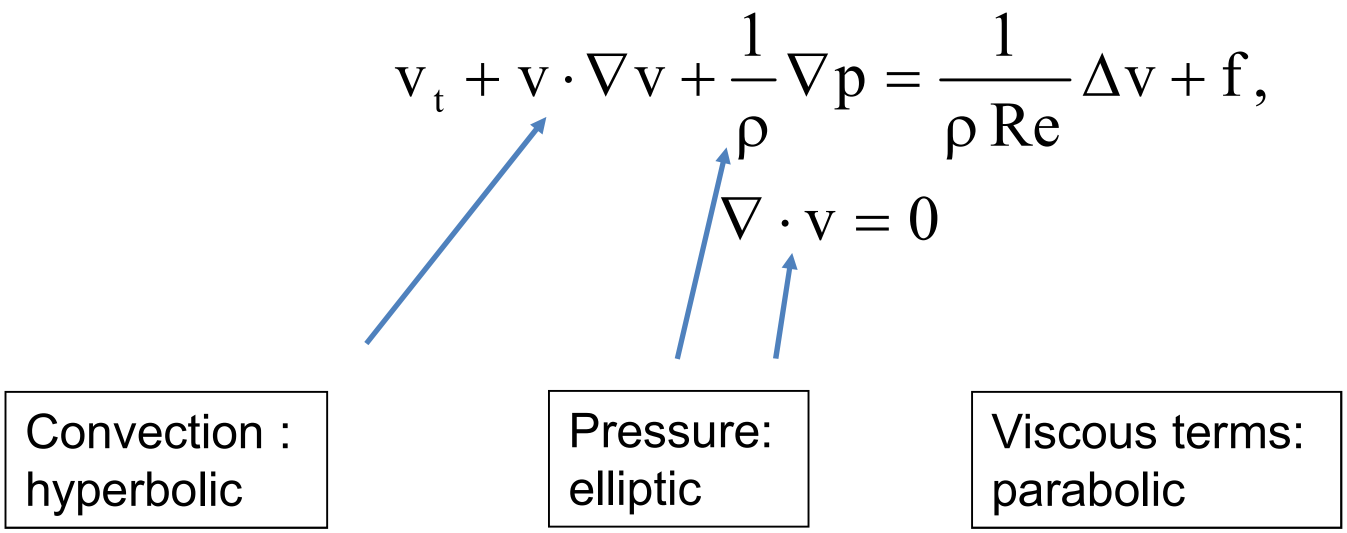
Properties of the Navier-Stokes Equation

- The Navier-Stokes equations are hyperbolic-parabolic equations.
- Problems:

  Small viscous terms and heat flux (high Re):

  hyperbolic terms dominate: Shocks and Rarefaction
- The numerical method must handle all phenomena: Shock capturing

Incompressible Navier-Stokes Equation - Type



$$v_t + v \cdot \nabla v + \frac{1}{\rho} \nabla p = \frac{1}{\rho\,\mathrm{Re}} \Delta v + f,$$

$$\nabla \cdot v = 0$$

Convection : hyperbolic

Pressure: elliptic

Viscous terms: parabolic

The incompressible Navier-Stokes equations are parabolic-elliptic. At high Reynolds numbers, the hyperbolic terms are dominating.

Numerical Schemes

- Numerical methods for solving partial differential equations
  1. Finite Difference (FD) methods
  2. Finite Element (FE) methods
  3. Finite Volume (FV) methods

Differences Methods (Finite Differences - FD)

- Idea:

$$\underset{\text{derivative}}{\underset{\uparrow}{u'(x)}} = \lim_{h \to 0} \underset{\text{difference quotient}}{\underset{\uparrow}{\frac{u(x+h) - u(x)}{h}}}$$

Exchange the derivatives by difference quotients.

Partial differential equation =>

System of differential equations are evaluated at grid points.

Step 1: Discretization of Computational Domain

Domain: Intervall $I = [a, b]$

Discretization: grid point $x_i = a + i\Delta x, \quad i = 0, \ldots, n$

Equidistant grid: grid spacing $\Delta x = \dfrac{b - a}{n}$

n+1: number of grid points

n: number of grid steps

Step 2: Difference Quotient

Taylor series

(1) $$u(x_{i-1}) = u(x_i) - \Delta x\, u_x(x_i) + \frac{\Delta x^2}{2} u_{xx}(x_i) - \frac{\Delta x^3}{6} u_{xxx}(x_i) + - \ldots$$

(2) $$u(x_{i+1}) = u(x_i) + \Delta x\, u_x(x_i) + \frac{\Delta x^2}{2} u_{xx}(x_i) + \frac{\Delta x^3}{6} u_{xxx}(x_i) + \ldots$$

For each derivative insert difference quotient

(2)- (1)

$$u(x_{i+1}) - u(x_{i-1}) = 2\Delta x u_x(x_i) + \frac{2\Delta x^3}{6} u_{xxx}(x_i) + O(\Delta x^5)$$

$$\frac{u(x_{i+1}) - u(x_{i-1})}{2\Delta x} = u_x(x_i) + \frac{\Delta x^2}{6} u_{xxx}(x_i) + O(\Delta x^4)$$

Step 2: Difference Quotient II

$$\boxed{\frac{u(x_{i+1})-u(x_{i-1})}{2\Delta x}=u_x(x_i)+O(\Delta x^2)}$$

Central difference quotient
2. order

$$\boxed{\frac{u(x_{i+1})-u(x_i)}{\Delta x}=u_x(x_i)+O(\Delta x)}$$

right-hand difference quotient
1. order

$$\boxed{\frac{u(x_i)-u(x_{i-1})}{\Delta x}=u_x(x_i)+O(\Delta x)}$$

left-hand difference quotient
1. order

$$\boxed{\frac{u(x_{i+1})-2u(x_i)+u(x_{i-1})}{h^2}=u_{xx}(x_i)+O(\Delta x^2)}$$

Central difference quotient ,
2. order for 2. derivative

- Difference quotient of higher order requires more than 3 points
- Difference quotient of different order and type
- Non-equidistant: terms are more complicate

FD Methods: Procedure

1. Step: Discretization of computational domain, grid
2. Step: Selection of difference quotient, derivatives are substituted by difference quotients
3. Step: Reordering of the difference equation
4. Step: Solving the difference equation linear system

FD - Poisson-Equation

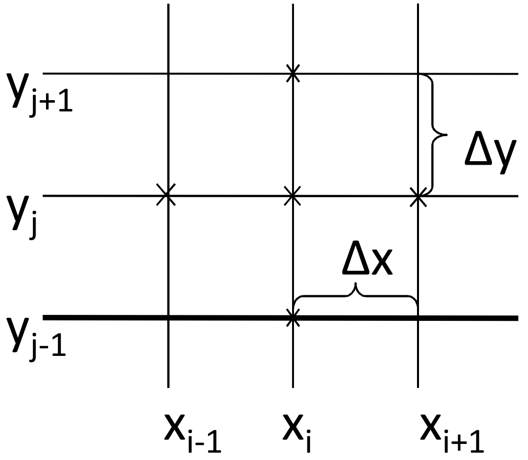
$$u_{xx} + u_{yy} = f$$

Domain : $[a, b] \times [c, d]$

Boundary value (Dirichlet):

$u(a,y)=u_a(y)$ for $x \in [a, b]$
$u(b,y)=u_b(y)$
$u(x,c)=u_c(x)$ for $y \in [c, d]$
$u(x,d)=u_d(x)$

1. Step: Discretization



equidistant grid spacing

$$\Delta x = \frac{b - a}{n_1}, \quad \Delta y = \frac{d - c}{n_2}$$

grid points

$$x_i = a + i\Delta x, \quad i = 0,\dots,n_1$$

$$y_j = c + j\Delta y, \quad j = 0,\dots,n_2$$

2. Step: Difference Quotient

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i\text{-}1,j}}{\Delta x^2} \quad \text{appr.} \quad u_{xx}(x_i, y_j)$$

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} \quad \text{appr.} \quad u_{yy}(x_i, y_j)$$

Insert the finite differences into differential equation of the Poisson equation:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i\text{-}1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = f_{i,j}$$

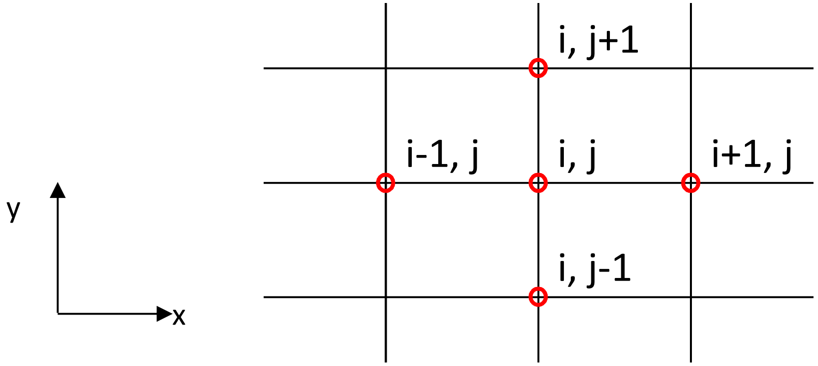
at each inner grid point x_i, y_j

3. Step: Rearranging of the Equations

Equation at each inner grid point

$$\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2}+\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}=f(x_i,y_j)$$

$$i=1,\dots,n_1\text{-}1\;,\; j=1,\dots,n_2-1$$



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$P_{i,j}=(x_i,y_j)$

$u_{i,j}\approx u(x_i,y_j)$

approximation

exact value

3. Step: Construction of Linear System

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = f(x_i, y_j)$$

$$i = 1, \dots, n_1 - 1 ,\ j = 1, \dots, n_2 - 1$$

Special treatment: boundary values

$$a_{i,j} u_{i,j-1} + b_{i,j} u_{i-1,j} + c_{i,j} u_{i,j} + d_{i,j} u_{i+1,j} + e_{i,j} u_{i,j+1} = \bar{f}_{i,j}$$

$$i = 1, \dots, n_1 - 1 ,\ j = 1, \dots, n_2 - 1$$

Form of Linear System

$$a_{i,j}u_{i,j-1} + b_{i,j}u_{i\text{-}1,j} + c_{i,j}u_{i,j} + d_{i,j}u_{i+1,j} + e_{i,j}u_{i,j+1} = \bar{f}_{i,j} \quad \text{für } \; i = 1, \ldots, n_1 \text{ -} 1 \, , \; j = 1, \ldots, n_2 - 1$$

$$u = \left(\begin{array}{c} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \\ \hline u_{12} \\ u_{22} \\ u_{32} \\ u_{42} \\ \hline u_{13} \\ u_{23} \\ u_{33} \\ u_{43} \\ \hline u_{14} \\ u_{24} \\ u_{34} \\ u_{44} \end{array} \right) \qquad A = \left(\begin{array}{cccc|cccc|cccc|cccc}
c_{11} & d_{11} & & & e_{11} & & & & & & & & & & & \\
b_{21} & c_{21} & d_{21} & & & e_{21} & & & & & & & & & & \\
& b_{31} & c_{31} & d_{31} & & & e_{31} & & & & & & & & & \\
& & b_{41} & c_{41} & & & & e_{41} & & & & & & & & \\ \hline
a_{12} & & & & c_{12} & d_{12} & & & e_{12} & & & & & & & \\
& a_{22} & & & b_{22} & c_{22} & d_{22} & & & e_{22} & & & & & & \\
& & a_{32} & & & b_{32} & c_{32} & d_{32} & & & e_{32} & & & & & \\
& & & a_{42} & & & b_{42} & c_{42} & & & & e_{42} & & & & \\ \hline
& & & & a_{13} & & & & c_{13} & d_{13} & & & e_{13} & & & \\
& & & & & a_{23} & & & b_{23} & c_{23} & d_{23} & & & e_{23} & & \\
& & & & & & a_{33} & & & b_{33} & c_{33} & d_{33} & & & e_{33} & \\
& & & & & & & a_{43} & & & b_{43} & c_{43} & & & & e_{43} \\ \hline
& & & & & & & & a_{14} & & & & c_{14} & d_{14} & & \\
& & & & & & & & & a_{24} & & & b_{24} & c_{24} & d_{24} & \\
& & & & & & & & & & a_{34} & & & b_{34} & c_{34} & d_{34} \\
& & & & & & & & & & & a_{44} & & & b_{44} & c_{44}
\end{array} \right)$$

Penta diagonal LS

Step 4: Solving the Linear System

- Numerical Methods

  Gauss-Algorithm

  Drawbacks: expensive, computes with all zero

  More suitable: Iterative methods

  Solving LS to a certain accuracy

Iterative Methods for Sparse Linear Systems

- **Classical iterative methods: Jacobi-, Gauß-Seidel-, SOR-Method** ill conditioned systems with small steps. Only for small systems.
- **CG-method (Method of conjungent gradients)** Matrices are symmetric and positive definite
- **Methods of general residual, Krylov-subspace methods** Generalization of the CG-Method: GMRES, BIGSTAB
- **Multigrid-methods** Surpasses the disadvantage of classical iterative methods by solving the equation on different fine grids. Fast, but depending on parameters.

Finite Differences for Heat Equation: Explicit

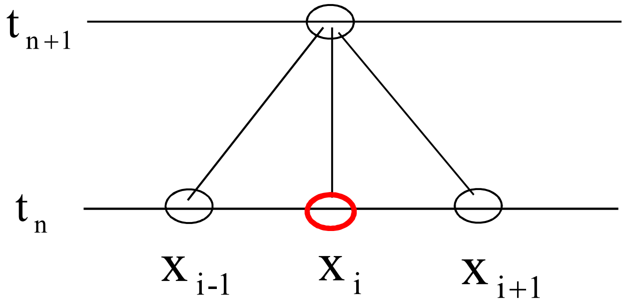
Heat equation: $$u_t = \kappa u_{xx}$$

explicit method $$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \kappa \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

forward central

differential quotient

Reordering for u_i^{n+1}: $$u_i^{n+1} = u_i^n + \frac{\kappa \Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$



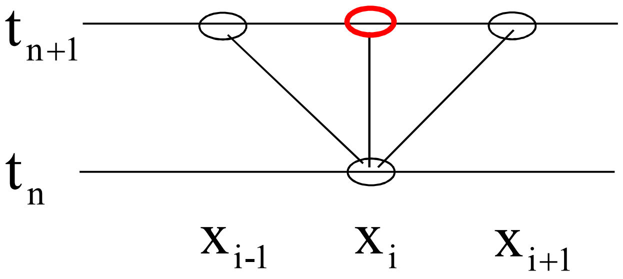
stencil

Finite Differences for Heat Equation: Implicit I

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \kappa \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$$

backward central

differential quotient



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Linear system of equations

Tridiagonal system of equations

$$-\frac{\alpha\Delta t}{\Delta x^2} u_{i-1}^{n+1} + \left(1 + 2\frac{\alpha\Delta t}{\Delta x^2}\right) u_i^{n+1} - \frac{\alpha\Delta t}{\Delta x^2} u_{i+1}^{n+1} = u_i^n, \quad i = 1, \ldots, n_1 - 1$$

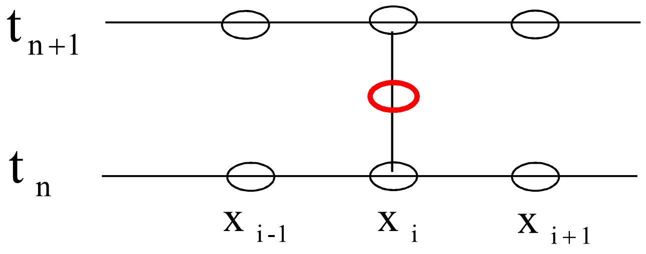
Implicit Methods

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \kappa \frac{u_{i+1}^{n+1} - 2u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x^2}$$

Fully implicit method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \kappa \frac{u_{i+1}^{n+1/2} - 2u_i^{n+1/2} - u_{i-1}^{n+1/2}}{\Delta x^2}$$

$$u_i^{n+1/2} = \frac{1}{2}\left(u_i^n + u_i^{n+1}\right)$$



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FD for Parabolic Differential Equations - Summary

Explicit methods

The explicit method O(Δt, Δx²) is conditionally stable and requires

$$\kappa \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2^d} \quad \Rightarrow \Delta t \sim \Delta x^2$$

The stability constrain leads to a practical accuracy of $O(\Delta x^2)$ which correspondence to a method of second order in space and time.

Implicit methods

The implicit method $O(\Delta t, \Delta x^2)$ or the Crank-Nicolson-method $O(\Delta t^2, \Delta x^2)$ are unconditionally stable.

FD for Hyperbolic Differential Equations

$$u_t + au_x = 0 \ , \quad a \in \Re$$

Linear system of transport equations

Explicit difference method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0 \qquad O\left(\Delta t, \Delta x^2\right)$$

or

$$u_i^{n+1} = u_i^n - a \frac{\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)$$

A Neumann stability analyze leads to:

Unconditionally unstable !!!!

One-Sided Differences

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+a\frac{u_{i+1}^n-u_i^n}{\Delta x}=0$$

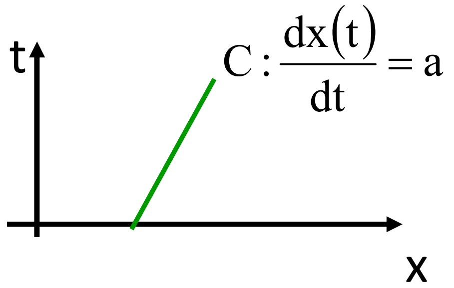
Left-hand differences

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+a\frac{u_i^n-u_{i-1}^n}{\Delta x}=0$$

Right-hand differences

Is this method stable?

Case a>0:



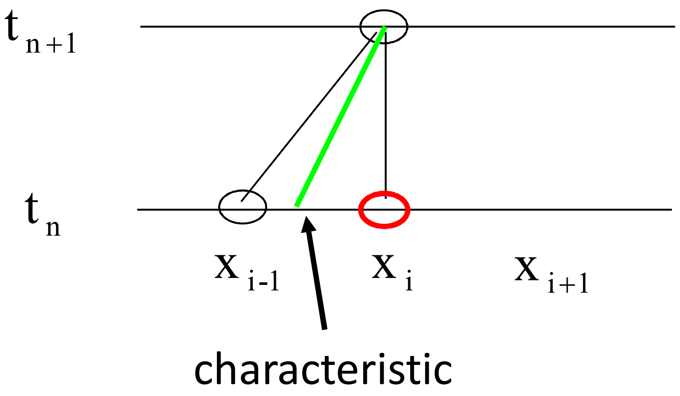
Dependency is motivated by physics!

Upwind Method or CIR-Scheme

von Neumann stability analysis: The CIR- method is conditionally stable under the condition:

$$|a|\frac{\Delta t}{\Delta x} < 1$$

CFL condition



The numerical dependency must match the physics.

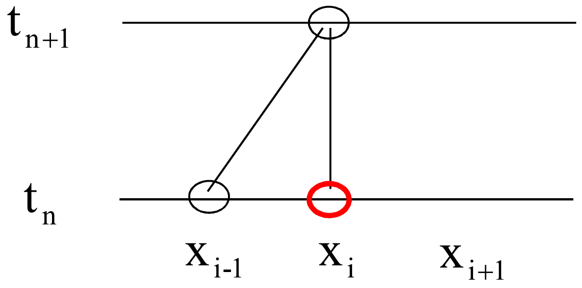
Upwind Method or CIR-Scheme II

$$u_i^{n+1} = u_i^n - a\frac{\Delta t}{\Delta x}\begin{cases} u_i^n - u_{i-1}^n & \text{für } \ a>0 \\ u_{i+1}^n - u_i^n & \text{für } \ a<0 \end{cases}$$

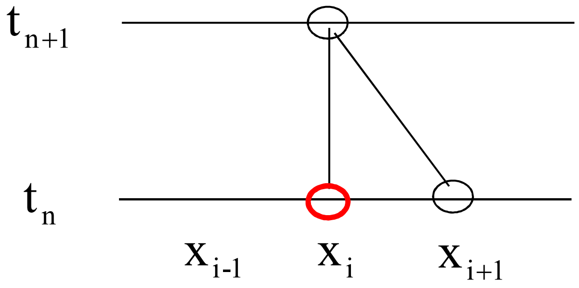
Courant, Isaacson, Rees - CIR

Upwind-method:

Design the differences in opposite direction to the information transport. The direction of the characteristic is reflected by the numerical method.



a>0



a<0

Implicit Method

Fully implicit

LS, unconditionally stable

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} = 0$$

Crank – Nicolson - Verfahren

LS, unconditionally stable

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^{n+1/2} - u_{i-1}^{n+1/2}}{2\Delta x} = 0$$

$$u_i^{n+1/2} = \frac{1}{2}\left(u_i^n + u_i^{n+1}\right)$$

Explicit Methods for Parabolic-Hyperbolic PDEs

Time step restriction

- Parabolic $\Delta t \sim \kappa \Delta x^2$

  Step size restriction: quadratic

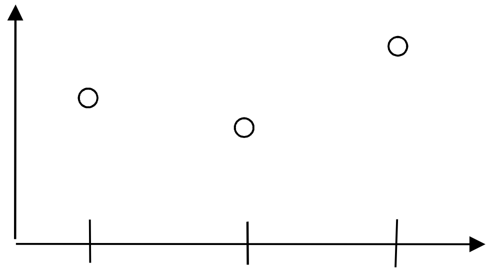
  Only possible for small κ
- Hyperbolic $\Delta t \sim a \Delta x$

  Step size restriction: linear

  Generally used for transient problems, stability requires Upwind approximation.

Summary FD

- Finite Differences (FD)
  - Solution is represented by point values
  - Simple coding, even for complex equations
  - Very complex on unstructured grids
  - Problem with discontinuities/strong gradients



Finite Element Method

Different approach: approximation by a continuous function:

$$\tilde{\mathrm{u}}(\mathrm{x}, \mathrm{t}) = \sum_{\mathrm{i}=1}^{\mathrm{N}} \hat{\mathrm{u}}_{\mathrm{i}}(t)\,\phi_i(x)$$

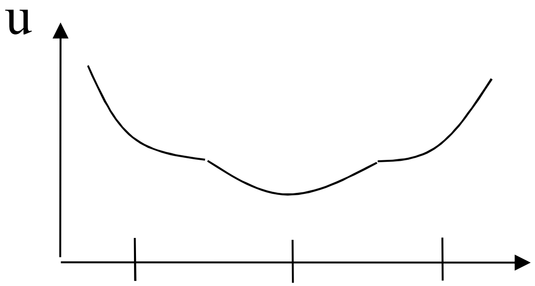
Degree of freedom (DOF)

Basis function

The degree of freedom are chosen in a manner to obtain the best possible approximation.

Summary FE

- Finite Element(FE)
  - Solution is represented by basis functions
  - Even more complex to code than FV
  - Arbitrary, unstructured grids
  - Special techniques at strong gradients

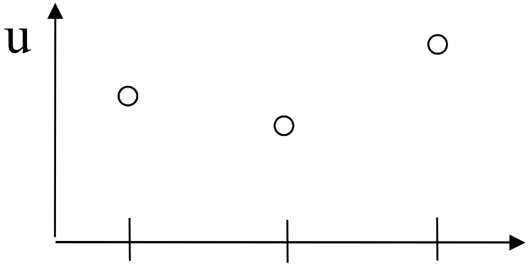


Finite Volume Method

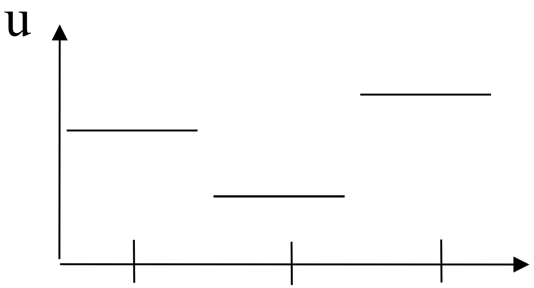
- Approximation of integral mean values
- Standard for conservation lows
- Strong gradients or discontinuities are easy to handle
- Flux calculation between adjoint cells: Riemann problem: shock capturing
- Reconstruction, piecewise polynomial

Comparison of FD, FE, FV

- Finite Difference (FD)
  - Solution is represented by point values
  - Simple coding, even for complex equations
  - Complex on unstructured grids
  - Problems at large gradients (Discontinuities)



- Finite Volume (FV)
  - Representation as cell mean value
  - More complex to code than FD
  - Arbitrary, unstructured grids
  - No problem at large gradients



- Finite Element (FE)
  - Solution is superposition of basis functions
  - More complex to code than FV
  - Easy on arbitrary, unstructured grids
  - Requires special techniques at large gradients and shocks