Finite Volume Methods

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1. Finite Volume (FV) Method for the Euler Equations
2. Godunov's Idea
3. Approximate Riemann Solvers
4. Finite Volume Method in 2D
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Euler Equations in One Dimensions

- The Euler eq. in 1D are defined as

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

- with the conserved variables **u** and the physical flux **f**(**u**)

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho v \\ e \end{pmatrix} \qquad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(e+p) \end{pmatrix}$$

- and the equation of state for a calorically perfect gas

$$p = (\gamma - 1)\rho\varepsilon \quad \Leftrightarrow \quad p = (\gamma - 1)\left(e - \tfrac{1}{2}\rho v^2\right)$$

Finite Volume Method in 1D

- The conservation law

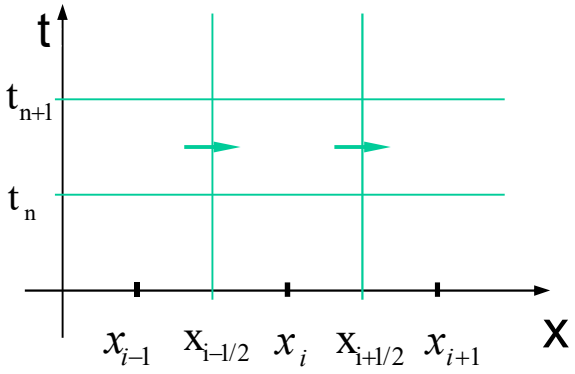
$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

- integrated over the control volume $[x_{i-½},x_{i+½}]x[t_n,t_{n+1}]$ yields

$$\int_{t_n}^{t_{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{u}_t(x,t)dxdt + \int_{t_n}^{t_{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{f}\big(\mathbf{u}(x,t)\big)_x dxdt = 0$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{u}(x,t_{n+1})dx - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{u}(x,t_n)dx +$$

$$\int_{t_n}^{t_{n+1}} \mathbf{f}\big(\mathbf{u}(x_{i+\frac{1}{2}},t)\big)dt - \int_{t_n}^{t_{n+1}} \mathbf{f}\big(\mathbf{u}(x_{i-\frac{1}{2}},t)\big)dt = 0$$



Finite Volume Method in 1D

- We consider temporal and spatial integral mean values

$$\underbrace{\frac{1}{\Delta t}\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\mathbf{u}(x,t_{n+1})dx}_{\mathbf{u}_i^{n+1}}-\underbrace{\frac{1}{\Delta t}\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\mathbf{u}(x,t_n)dx}_{\mathbf{u}_i^{n}}+$$

$$\frac{1}{\Delta x}\underbrace{\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\mathbf{f}\big(\mathbf{u}(x_{i+\frac{1}{2}},t)\big)dt}_{\mathbf{f}_{i+\frac{1}{2}}}-\frac{1}{\Delta x}\underbrace{\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\mathbf{f}\big(\mathbf{u}(x_{i-\frac{1}{2}},t)\big)dt}_{\mathbf{f}_{i-\frac{1}{2}}}=0$$

- The discrete form is

$$\mathbf{u}_i^{n+1}=\mathbf{u}_i^n-\frac{\Delta t}{\Delta x}\big(\mathbf{g}_{i+\frac{1}{2}}^n-\mathbf{g}_{i-\frac{1}{2}}^n\big)$$

Finite Volume Method in 1D

- The discrete form of the FV method in 1D is given by

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{g}_{i+\frac{1}{2}}^n - \mathbf{g}_{i-\frac{1}{2}}^n\right)$$

- Integral mean value: $\mathbf{u}_i^n = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{u}(x, t_n)\,dx$

- Numerical flux: $\mathbf{g}_{i+\frac{1}{2}} :\approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{f}\left(\mathbf{u}(x_{i+\frac{1}{2}}, t)\right)dt$

- Problem: Computation is performed on cell mean values, but the values at the cell edges are required for the flux calculation. How is the numerical flux computed?

Flux Calculation: Central Flux

- Compute mean value of both fluxes at the cell edges:

$$\mathbf{g}_{i+\frac{1}{2}} := \frac{1}{2}\left(\mathbf{f}\big(\mathbf{u}(x_i,t)\big) + \mathbf{f}\big(\mathbf{u}(x_{i+1},t)\big)\right)$$

- Formula is identical to FV with a central differential quotient, which is unconditionally unstable:

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{2\Delta x}\left(\mathbf{f}\big(\mathbf{u}(x_{i+1},t)\big) - \mathbf{f}\big(\mathbf{u}(x_{i-1},t)\big)\right).$$

- Can be stabilized by additional artificial viscosity --> Jameson-Schmidt-Turkel scheme

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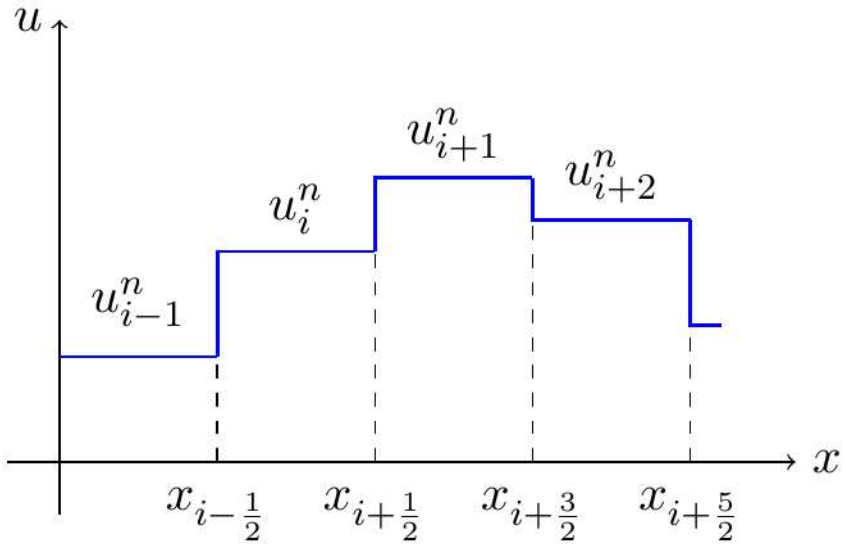
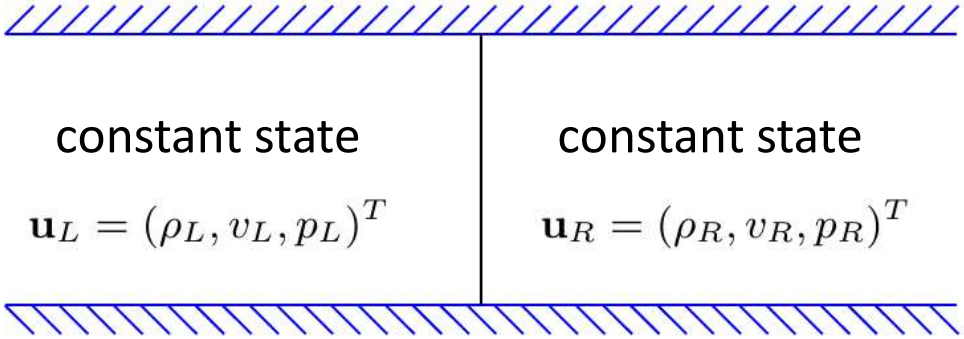
Godunov‘s Idea in 1D – Godunov‘s method

- Piecewise constant values in each cell.
- Solve Riemann problem at each cell interface.
- Exact solution of the non-linear Riemann problem.



Godunov‘s Idea in 1D – Godunov‘s method

- At each cell edge, a shock tube problem is considered:

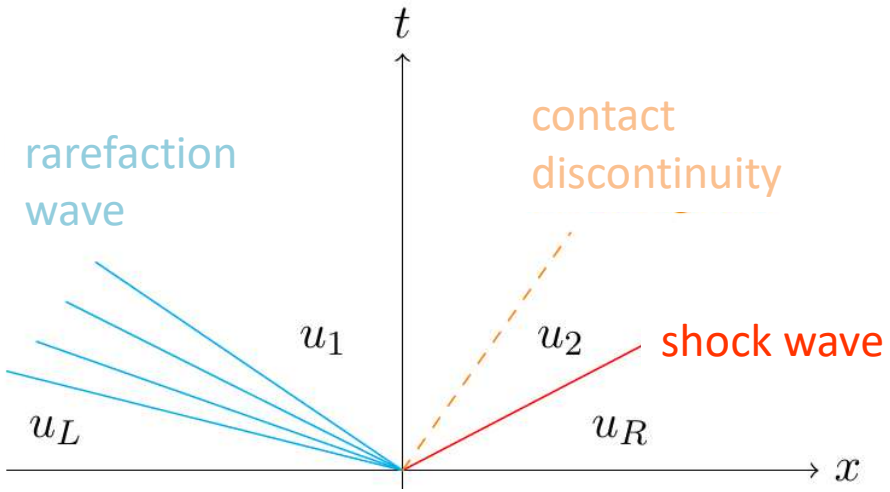


$$\mathbf{u}(x, t = 0) = \begin{cases} \mathbf{u}_L & : x < 0 \\ \mathbf{u}_R & : x \geq 0 \end{cases}$$

Riemann Problem for the Euler equations

- At each cell edge, the Riemann problem is solved:

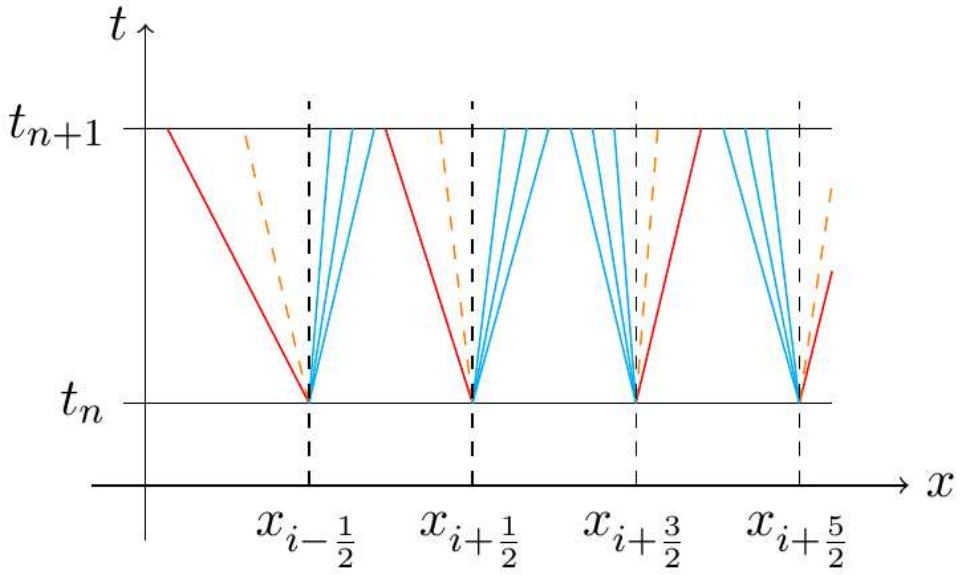
$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \qquad \mathbf{u}(x, t=0) = \begin{cases} \mathbf{u}_L & : x < 0 \\ \mathbf{u}_R & : x \geq 0 \end{cases}$$



- The exact solution is obtained via fix-point iteration.

Formulation as Finite Volume Method

- FV method: $\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}(\mathbf{g}_{i+\frac{1}{2}}^n - \mathbf{g}_{i-\frac{1}{2}}^n)$
- Numerical flux of Godunov at $x_{i+1/2}$: $\mathbf{g}_{i+\frac{1}{2}}^n = \mathbf{f}(\mathbf{u}_{RP}(\mathbf{u}_i, \mathbf{u}_{i+1}, \mathbf{n}))$
- u_{RP}: Solution of the Riemann problem at the cell interfaces.
- Drawback:
  - expensive



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Approximate Riemann Solvers

Can be classified as follows:

1. Flux-difference splitting schemes (Godunov-type schemes)

--> Godunov, Roe, HLL, Lax-Friedrichs, Osher

2. Flux-vector splitting schemes

--> Steger-Warming, van Leer, AUSM, AUSMD, …

Approximate Riemann Solvers

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Godunov-type flux have to fulfill the following:

1. Consistency with the integral conservation
2. Consistency with the entropy inequality eq.
3. Consistency

Roe Riemann Solver

- P. Roe (1981) exchanges the exact solution of the Riemann problem by the solution of the linearized Riemann problem:

$$\mathbf{u}_t + \mathbb{A}_{\mathrm{LR}}\mathbf{u}_x = 0, \quad \mathbf{u}(x, t=0) = \begin{cases} \mathbf{u}_L & x < 0 \\ \mathbf{u}_R & x \geq 0 \end{cases}$$

- The matrix $\mathbb{A}_{\mathrm{LR}} = \mathbb{A}_{\mathrm{LR}}(\mathbf{u}_L, \mathbf{u}_R)$ is called Roe-matrix, if
  1. Consistency: $\mathbb{A}_{\mathrm{LR}}(\mathbf{u}, \mathbf{u}) = \mathbb{A}(\mathbf{u})$
  2. $\mathbb{A}_{\mathrm{LR}}$ is diagonalizable (hyperbolic)
  3. $\mathbb{A}_{\mathrm{LR}}$ is consistent with the integral conservation:

$$\mathbf{f}(\mathbf{u}_R) - \mathbf{f}(\mathbf{u}_L) = \mathbb{A}_{\mathrm{LR}}(\mathbf{u}_R - \mathbf{u}_L)$$

Derivation of the Roe Mean Values

- Flux and Jacobian of the 1D Euler equations

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{u}_t + \mathbb{A}(\mathbf{u})\mathbf{u}_x = 0, \qquad \mathbb{A}(\mathbf{u}) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$$

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(e+p) \end{pmatrix}, \quad \mathbb{A}(\mathbf{u}) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}v^2 & (3-\gamma)v & \gamma-1 \\ v\left(\frac{\gamma-1}{2}v^2 - H\right) & H-(\gamma-1)v^2 & \gamma v \end{pmatrix}$$

- Enthalpy $H = \frac{e+p}{\rho}$, speed of sound $c^2 = \gamma\frac{p}{\rho}, \quad p = (\gamma-1)\left(e - \frac{1}{2}\rho v^2\right)$
- The eigenvalues are: $a_1 = v - c, a_2 = v, a_3 = v + c$

Roe Mean Values for Ideal Gas

- The Roe-Matrix is given by

$$\mathbb{A}_{\mathrm{LR}}(\mathbf{u}_L, \mathbf{u}_R) = \mathbb{A}(\bar{\mathbf{u}})$$

- with the Roe mean values for calorically perfect gas:

$$\bar{\rho} = \sqrt{\rho_R \rho_L},$$

$$\bar{v} = \frac{\sqrt{\rho_R} v_R + \sqrt{\rho_L} v_L}{\sqrt{\rho_R} + \sqrt{\rho_L}},$$

$$\bar{H} = \frac{\sqrt{\rho_R} H_R + \sqrt{\rho_L} H_L}{\sqrt{\rho_R} + \sqrt{\rho_L}},$$

$$\bar{c}^2 = (\gamma - 1)\left(\bar{H} - \frac{1}{2}\bar{v}^2\right).$$

What is the Reason for the Linearization?

- If a matrix $\mathbb{A}$ is diagonalizable, a Matrix $\mathbf{\Lambda}$ exists which satisfies

$$\mathbf{\Lambda} = \mathbb{R}^{-1}\mathbb{A}\mathbb{R} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = \mathrm{diag}(a_1, a_2, a_3)$$

- Matrix of eigenvectors: $\mathbb{R} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$
- $\mathbf{r}_i$ – eigenvectors for the eigenvalue a_i,i.e., $\mathbb{A}\mathbf{r}_i = a_i\mathbf{r}_i$

What is the Reason for the Linearization?

- The linearized eq. system can be transformed into the characteristic normal form:

$$\mathbf{u}_t + \mathbb{A}_{LR}\mathbf{u}_x = 0$$
$$\mathbb{R}^{-1}\mathbf{u}_t + \mathbb{R}^{-1}\mathbb{A}_{LR}\mathbf{u}_x = 0$$
$$\underbrace{\mathbb{R}^{-1}\mathbf{u}_t}_{\mathbf{w}_t} + \underbrace{\mathbb{R}^{-1}\mathbb{A}_{LR}\mathbb{R}}_{\mathbf{\Lambda}}\underbrace{\mathbb{R}^{-1}\mathbf{u}_x}_{\mathbf{w}_x} = 0$$
$$\mathbf{w}_t + \mathbf{\Lambda}\mathbf{w}_x = 0$$

- Characteristic variable: $\mathbf{w} = \mathbb{R}^{-1}\mathbf{u}$
- m decoupled scalar linear transport equations

Exact Solution of the Linearized Riemann Problem for the Euler Eq.

1. Transformation into the characteristic variables

$$\mathbf{u}_R - \mathbf{u}_L = \mathbb{R}(\mathbf{w}_R - \mathbf{w}_L) = \sum_{k=1}^{3} \gamma_k \mathbf{r}_k$$

2. Determine the coefficients, i.e., the Roe mean values,

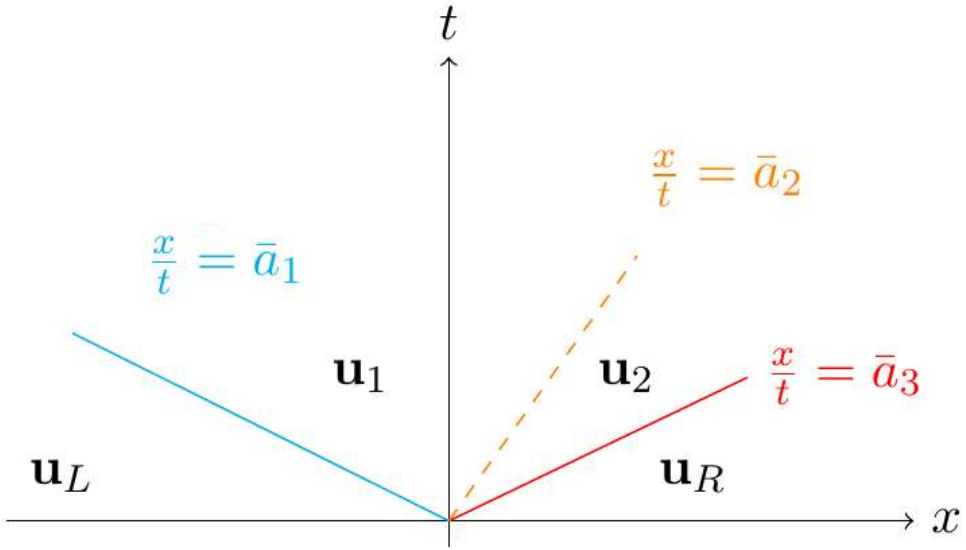
$$\gamma_1 = -\frac{1}{2\bar{c}}\left[\Delta m_1 - \Delta\rho(\bar{v}_1 + \bar{c})\right] - \frac{1}{2}\gamma_2$$

$$\gamma_2 = -\frac{\gamma - 1}{\bar{c}^2}\left[\Delta\rho(\bar{v}_1^2 - H) - \bar{v}_1 \Delta m_1 + \Delta\bar{e}\right]$$

$$\gamma_3 = \Delta\rho - \gamma_1 - \gamma_2$$

Exact Solution of the Linearized Riemann Problem for the Euler Eq. II

3. The four constant states are



$$\mathbf{u}_1 = \mathbf{u}_L + \gamma_1 \mathbf{r}_1$$
$$\mathbf{u}_2 = \mathbf{u}_L + \gamma_1 \mathbf{r}_1 + \gamma_2 \mathbf{r}_2$$

correction according to wave propagation; dissipation matrix

Numerical flux: $$\mathbf{g}_{\text{Roe}}(\mathbf{u}_L, \mathbf{u}_R) = \frac{1}{2}(\mathbf{f}(\mathbf{u}_L) + \mathbf{f}(\mathbf{u}_R)) - \frac{1}{2}\sum_{k=1}^{3} |a_k| \gamma_k \mathbf{r}_k$$

central flux

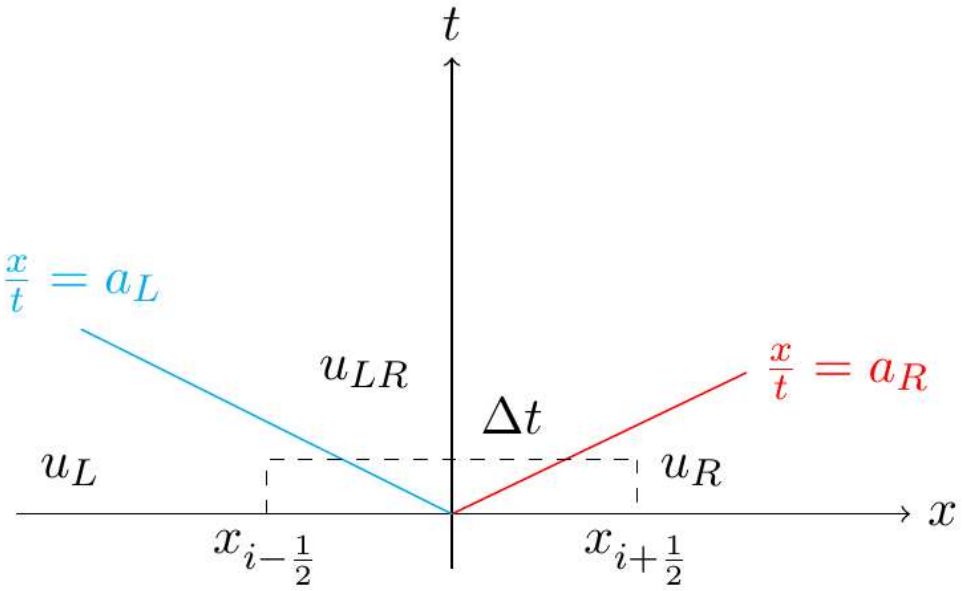
Properties of the Roe Scheme

- Exact resolution of the shock wave and the contact discontinuity.
- Problem: The Roe method violates the consistency of the entropy condition at rarefaction fans.
- This is only a problem for transient rarefaction waves.
- Solution: An entropy-fix ensures consistency with the entropy inequality eq., e.g., the approach of Harten redefines the wave speeds

$$\tilde{a}_{1,3} = \begin{cases} \bar{a}_{1,3} & : |a_{1,3}| > \delta,\ \delta \in \mathbb{R},\ \delta > 0 \\ \delta & : \text{ else} \end{cases}$$

Riemann Solver of Harten, Lax & Van Leer (HLL)

- The HLL method is the simplest Godunov-type Riemann solver.
- Only the fastest and the slowest waves are considered.



Riemann-Solver of Harten, Lax & van Leer II

- Additional simplification of the Roe solver, i.e., an approximation of the linearized Riemann problem of Roe.
- Only one average state is present:

$$\tilde{\mathbf{w}}\left(\frac{x}{t}, \mathbf{u}_L, \mathbf{u}_R\right) = \begin{cases} \mathbf{u}_L & : \frac{x}{t} < a_L \\ \mathbf{u}_{LR} & : a_L \leq \frac{x}{t} \leq a_R \\ \mathbf{u}_R & : \frac{x}{t} > a_R \end{cases}$$

- a_L, a_R are the smallest and largest wave speeds
- u_{LR} is the average state

Riemann Solver of Harten, Lax & van Leer III

- Consistency with the integral conservation yields the average state

$$\mathbf{u}_{LR} = \frac{a_R \mathbf{u}_R - a_L \mathbf{u}_L - \mathbf{f}(\mathbf{u}_R) + \mathbf{f}(\mathbf{u}_L)}{a_R - a_L}.$$

- The numerical flux function is given by

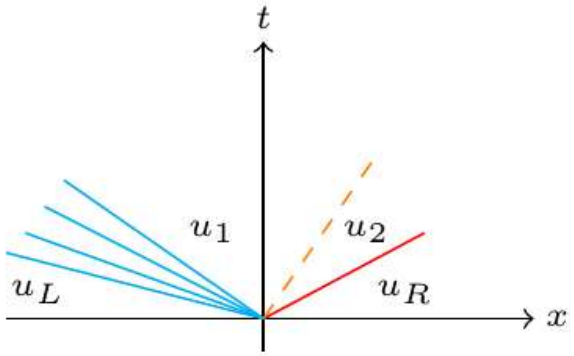
$$\mathbf{g}_{\mathrm{HLL}}(\mathbf{u}_L, \mathbf{u}_R) = \frac{a^+ \mathbf{f}(\mathbf{u}_L) - a^- \mathbf{f}(\mathbf{u}_R)}{a^+ - a^-} + \frac{a^+ a^-}{a^+ - a^-}(\mathbf{u}_R - \mathbf{u}_L)$$

- The wave speeds are computed as

$$a^- = \min\left(0, v_L - c_L, \bar{v} - \bar{c}\right) \qquad a^+ = \max\left(0, v_R + c_R, \bar{v} + \bar{c}\right)$$

- Local Lax-Friedrichs: use only the (locally) fastest wave speed

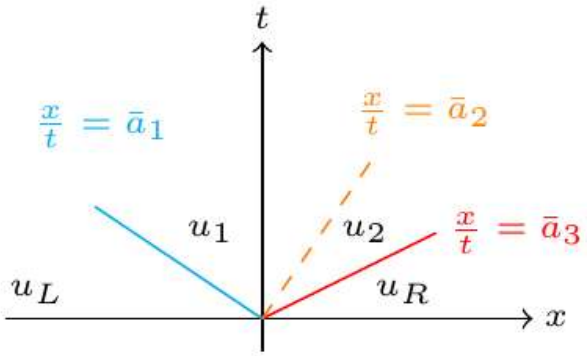
Riemann Solvers - Summary



Godunov scheme

Exact solution of the non-linear Riemann problem;

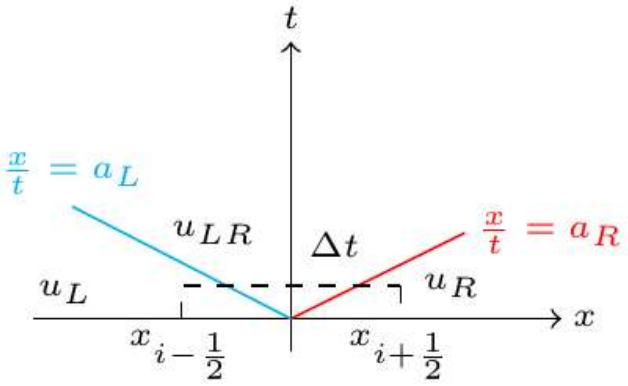
iterative



Roe scheme

Exact solution of the linearized Riemann problem;

characteristic theory



HLL scheme

A priori estimation of the fastest and slowest wave velocities

Flux Vector Splitting Method

- Flux-vector splitting methods split the flux into a right and left travelling part: $\mathbf{f}(\mathbf{u}) = \underbrace{\mathbf{f}^+(\mathbf{u})}_{\text{left}} + \underbrace{\mathbf{f}^-(\mathbf{u})}_{\text{right}}$
- with non-negative eigenvalues $\mathbb{A}^+(\mathbf{u}) = \dfrac{d\mathbf{f}^+(\mathbf{u})}{d\mathbf{u}}$
- and non-positive eigenvalues $\mathbb{A}^-(\mathbf{u}) = \dfrac{d\mathbf{f}^-(\mathbf{u})}{d\mathbf{u}}$
- The numerical flux is given as

$$\mathbf{g}(\mathbf{u}_L, \mathbf{u}_R) = \mathbf{f}^+(\mathbf{u}_L) + \mathbf{f}^-(\mathbf{u}_R)$$

Flux Vector Splitting: Steger & Warming

1. Diagonalization

$$\mathbf{\Lambda} = \mathbb{R}^{-1}\mathbb{A}\mathbb{R} \text{ mit } \mathbf{\Lambda} := \mathrm{diag}\left(v - c, v, v + c\right)$$

2. Splitting of the eigenvalues

$$\mathbf{\Lambda} = \mathbf{\Lambda}^{+} + \mathbf{\Lambda}^{-} \text{ mit}$$
$$\mathbf{\Lambda}^{+} := \mathrm{diag}\left(a_1^{+}, a_2^{+}, a_3^{+}\right) \text{ und } \mathbf{\Lambda}^{-} := \mathrm{diag}\left(a_1^{-}, a_2^{-}, a_3^{-}\right)$$

3. Split the flux: $\mathbf{g}(\mathbf{u}_L, \mathbf{u}_R) = \mathbf{f}^{+}(\mathbf{u}_L) + \mathbf{f}^{-}(\mathbf{u}_R)$

- Using the Euler Theorem: $\mathbf{f}(\mathbf{u}) = \mathbb{A}(\mathbf{u})\mathbf{u},\ \mathbb{A}(\mathbf{u}) := \dfrac{d\mathbf{f}(\mathbf{u})}{d\mathbf{u}}.$

- This yields the numerical flux function of Steger & Warming

$$\mathbf{g}(\mathbf{u}_L, \mathbf{u}_R) = \mathbf{f}^{+}(\mathbf{u}_L) + \mathbf{f}^{-}(\mathbf{u}_R) = \mathbb{A}^{+}(\mathbf{u}_L)\mathbf{u}_L + \mathbb{A}^{-}(\mathbf{u}_R)\mathbf{u}_R$$

Flux-Vector Splitting: Steger & Warming II

$f^{\pm}_{1,1}$	$\frac{\rho}{2\gamma}\left(2(\gamma-1)\cdot a^{\pm}_2 + a^{\pm}_1 + a^{\pm}_4\right)$
$f^{\pm}_{1,2}$	$f^{\pm}_{1,1}\cdot v_1 + \left(a^{\pm}_4 - a^{\pm}_1\right)\cdot\frac{\rho c}{2\gamma}$
$f^{\pm}_{1,3}$	$f^{\pm}_{1,1}\cdot v_2$
$f^{\pm}_{1,4}$	$f^{\pm}_{1,1}\cdot\frac{v_1^{\,2}+v_2^{\,2}}{2} + \left(a^{\pm}_4 - a^{\pm}_1\right)\cdot\frac{\rho c v_1}{2\gamma} + \left(a^{\pm}_4 + a^{\pm}_1\right)\cdot\frac{\rho c^2}{2(\gamma-1)\gamma}$

Steger-Warming Flux in x-direction

Flux-Vector Splitting: van Leer in x-direction

Mach number M	Right going flux	Left going flux
$M \geq 1$	$F_1^+ = F_1$	$F_1^- = 0$
$-1 < M < 1$	$F_{1,1}^+ = \frac{\rho c}{4}(M+1)^2, M = \frac{v_1}{c}$ $F_{1,2}^+ = F_{1,1}^+ \cdot \frac{C_1}{\gamma}$ $F_{1,3}^+ = F_{1,1}^+ \cdot v_2$ $F_{1,4}^+ = F_{1,2}^+ \cdot C_1 \cdot \frac{\gamma}{2(\gamma^2-1)} + \frac{1}{2}F_{1,3}^{\pm} \cdot v_2$ mit $C_1 = (\gamma - 1) \cdot v_1 + 2c$	$F_{1,1}^- = -\frac{\rho c}{4}(1-M)^2, M = \frac{v_1}{c}$ $F_{1,2}^- = F_{1,1}^- \cdot \frac{C_1}{\gamma}$ $F_{1,3}^- = F_{1,1}^- \cdot v_2$ $F_{1,4}^- = F_{1,2}^- \cdot C_1 \cdot \frac{\gamma}{2(\gamma^2-1)} + \frac{1}{2}F_{1,3}^- \cdot v_2$ mit $C_1 = (\gamma - 1) \cdot v_1 - 2c$
$M \leq -1$	$F_1^+ = 0$	$F_1^- = F_1$

Flux Computation - Summary

- Flux computation is upwind based:

1. Flux-difference splitting schemes (Godunov-type schemes)

   --> Godunov, Roe, HLL, Lax-Friedrichs, Osher

2. Flux-vector splitting schemes

   --> Steger-Warming, van Leer, AUSM, AUSMD, …

- Assumes piecewise constant cell values

  --> first order accurate; results are quite similar

Content

Finite Volume Method in 2D

- The differential equation

$$\mathbf{U}_t + \nabla \cdot \mathbb{F}^C(\mathbf{U}) = 0$$

- requires the differentiability of the solution, which cannot be guaranteed.
- In the context of the Euler eq., these are shock waves and contact discontinuities.

Finite Volume Method in 2D II

- The integral equation, in turn, has no restriction to the differentiability of the solution:

$$\int_{\mathbf{V}} \mathbf{U}_t \; d\mathbf{x} + \int_{\mathbf{V}} \nabla \cdot \mathbb{F}^C(\mathbf{U}) \; d\mathbf{x} = 0, \; \mathbf{x} = [x, y]^T$$

- Gauss's theorem converts the volume integral into a surface integral

$$\int_{\mathbf{V}} \mathbf{U}_t \, d\mathbf{x} + \oint_{\partial \mathbf{V}_i} \mathbb{F}^C(\mathbf{U}_{RP}) \cdot \mathbf{n} \, dS = 0.$$

Finite Volume Method in 2D II

- An essential step for the construction of the numerical scheme is to consider integral mean values in each cell:

$$\int_{\mathbf{V}} \mathbf{U}_t \, d\mathbf{x} = \mathbf{V}_i \bar{\mathbf{U}}_{i,t} = \mathbf{V}_i \mathbf{U}_{i,t}.$$

- This leads to

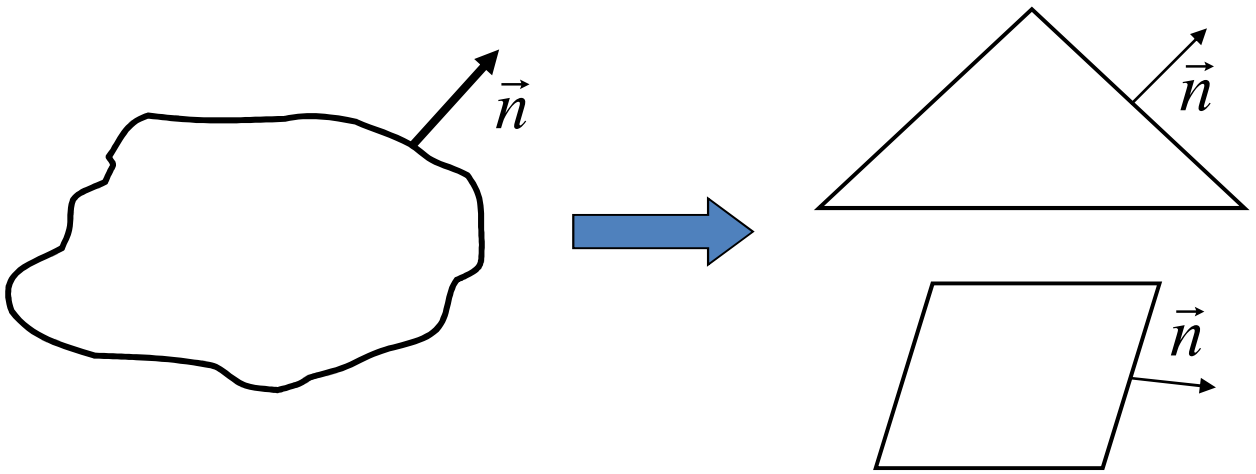
$$\mathbf{V}_i \, \mathbf{U}_{i,t} + \oint_{\partial \mathbf{V}_i} \mathbb{F}^C(\mathbf{U}_{RP}) \cdot \mathbf{n} \, dS = 0.$$

- Thus, the spatial operator is defined as

$$\mathbf{U}_{i,t} = R_i = R(\mathbf{U}_i) = -\frac{1}{\mathbf{V}_i} \oint_{\partial \mathbf{V}_i} \mathbb{F}^C(\mathbf{U}_{RP}) \cdot \mathbf{n} \, dS.$$

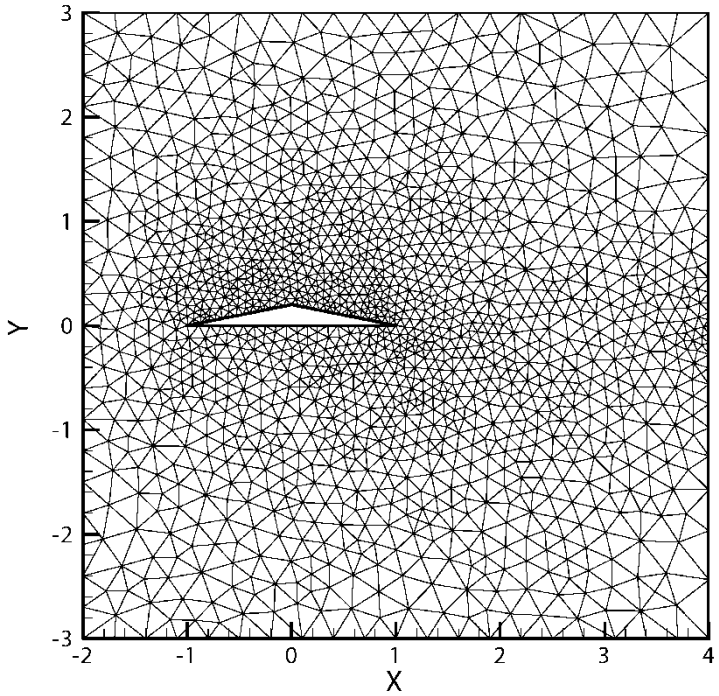
Control Volume

- To construct a numerical scheme, the evaluation of the surface integral is simplified by requiring piecewise constant edges of the control volume.



Computational Grid

- The whole domain is split into N overlapping tetrahedral elements.



Solving the Surface Integral

- Using a simplified geometry, the surface integral can be replaced by a sum over all edges

$$R_i = -\frac{1}{\mathbf{V}_i} \sum_{e_{ij} \subset \partial \mathbf{V}_i} \oint_{e_{ij}} \mathbb{F}^C(\mathbf{U}_{RP}) \cdot \mathbf{n} \, dS_j.$$

- The integration is performed via a numerical integration, e.g., Gaussian integration. The integration point is located at the side center, this is sufficiently accurate for 1. and 2. order

$$R_i \approx -\frac{1}{\mathbf{V}_i} \sum_{e_{ij}} |e_{ij}| \, \mathbb{F}^C(\mathbf{U}_{ij,RP}) \cdot \mathbf{n}_{ij}.$$

- $e_{ij:}$ cell edges of the cell $\boldsymbol{V}_i$

Reduction to 1D Riemann Problems: Rotation

- The Euler eq. are rotationally invariant: $\mathbb{F}^C(\mathbf{U}) \cdot \mathbf{n} = \mathbb{T}^{-1}\mathbb{F}^C(\mathbb{T}\mathbf{U})$
- The rotational matrix T is defined as

$$\mathbb{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n_1 & n_2 & 0 \\ 0 & -n_2 & n_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbb{T}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n_1 & -n_2 & 0 \\ 0 & n_2 & n_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

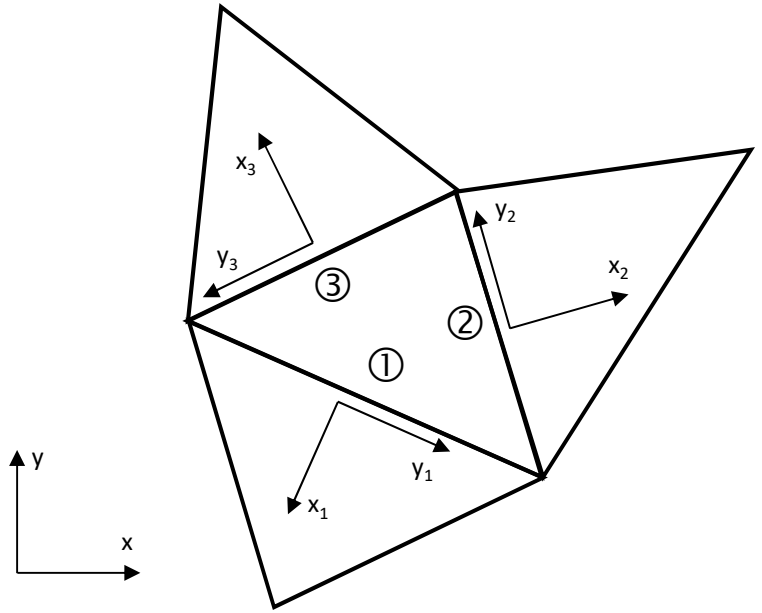
- Replacing the phys. by the num. flux: $\mathbb{F}^C(\mathbf{U}_{RP}) \cdot \mathbf{n} \approx g(\mathbf{U}_L, \mathbf{U}_R; \mathbf{n})$
- Thus, the residual is given by

$$\mathbf{U}_{i,t} = R_i \approx -\frac{1}{V_i} \sum_{e_{ij}} |e_{ij}| \, \mathbb{T}^{-1} g\left(\mathbb{T}\mathbf{U}_i, \mathbb{T}\mathbf{U}_j; [1,0]^{\mathrm{T}}\right).$$

Flux Computation in 2D - Summary

Steps of the flux computation:

1. Rotating the state at each cell into the local coordinate system
2. Computing the 1D Riemann Problem at each edge
3. Rotating the fluxes back into the global coordinate system
4. Computing the residual for each cell



$$\mathbf{U}_{i,t} = R_i \approx -\frac{1}{V_i} \sum_{e_{ij}} |e_{ij}| \, \mathbb{T}^{-1} g\left(\mathbb{T}\mathbf{U}_i, \mathbb{T}\mathbf{U}_j; [1,0]^{\mathrm{T}}\right).$$

Content

Temporal Discretization

- After the computation of the residuum, the temporal integral has to be solved.
- Following the method-of-lines approach, the temporal and spatial integration can be seperated

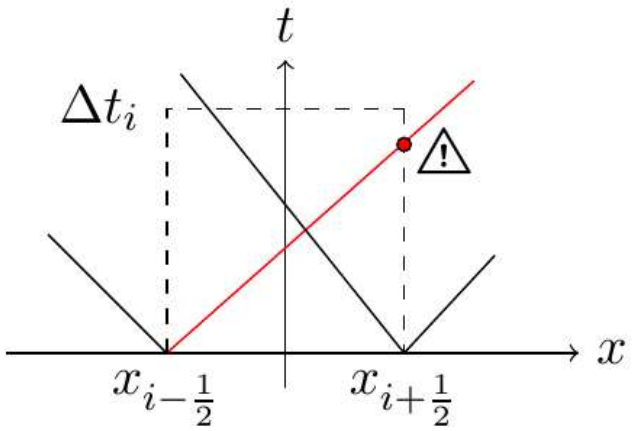
$$\mathbf{U}_{i,t} = R_i \approx -\frac{1}{\mathbf{V}_i} \sum_{e_{ij}} |e_{ij}| \, \mathbb{F}^C(\mathbf{U}_{ij,RP}) \cdot \mathbf{n}_{ij}.$$

- For first order methods, the left-hand Riemann sum results in

$$\frac{\mathbf{U}_i^{n+1} - \mathbf{U}_i^n}{\Delta t} = -\frac{1}{\mathbf{V}_i} \sum_{e_{ij}} |e_{ij}| \, \mathbb{F}^C(\mathbf{U}_{ij,RP}) \cdot \mathbf{n}_{ij}.$$

Time Step Size - the CFL Condition

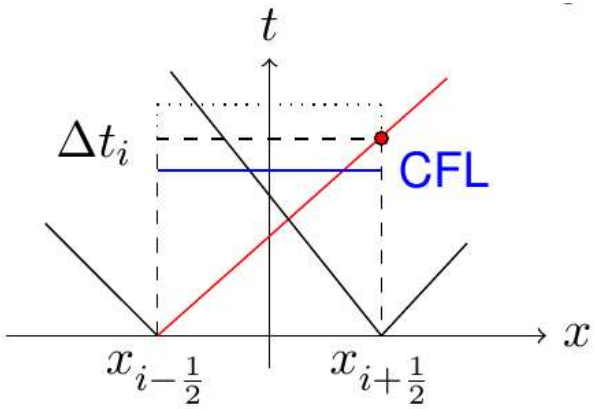
- An explicit time discretization is conditional stable.
- The time step restriction can be visualized in 1D:



- The flux computation requires a constant state at the cell edges. This is only satisfied, if no wave reaches the next cell during one step.

Determination of the Maximal Time Step

- The maximal time step is limited by the time of the fastest wave propagation through the cell:



$$\frac{\Delta t_i}{\Delta x_i} = \frac{1}{|v_i| + c_i}$$

$$\Longrightarrow \Delta t_i = \frac{\Delta x_i}{|v_i| + c_i}$$

with $\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$

- In practice, the time step is multiplied by a factor, the CFL number (named after Courant, Friedrichs und Levy), thus

$$\Delta t_i = \mathrm{CFL}\frac{\Delta x_i}{|v_i| + c_i}$$

Impact of the Discretization on the Time Step

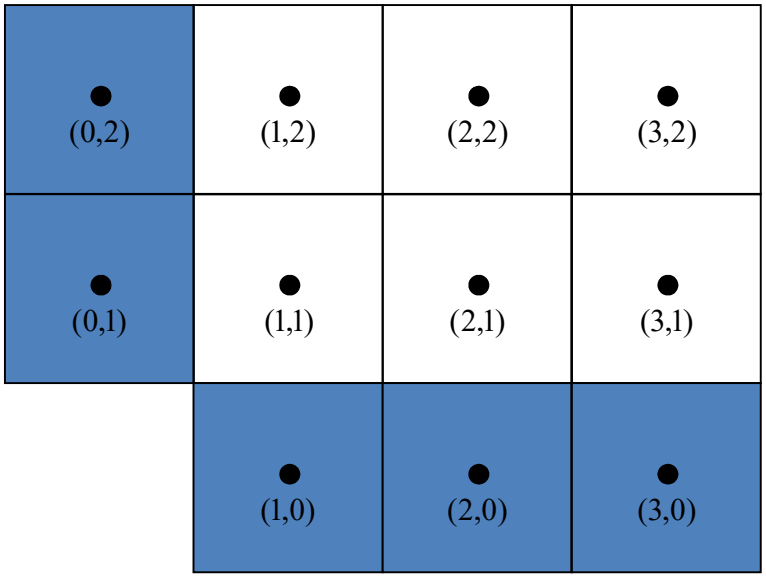
- The discretization influences the maximal possible time step.
- The order of the spatial discretization determines the order of the temporal integration.
- For higher dimensions, the time step is reduced by multi-dimensional effects.
- The maximal CFL number depends on the chosen time and spatial integration method and is defined by numerical experiments or given in tables.

Content

1. Finite Volume (FV) method for the Euler equations
2. Godunov's idea: the shock tube
3. Approximate Riemann solvers
4. Finite Volume method in 2D
5. Temporal Discretization
6. Boundary Conditions

Boundary Conditions: Ghost Cells

- Ghost cells are used for boundary conditions.
- The BC value is inserted into the flux computation, enforcing the boundary conditions weakly.

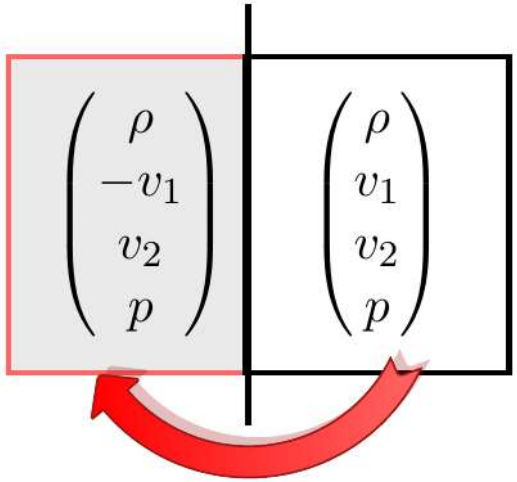


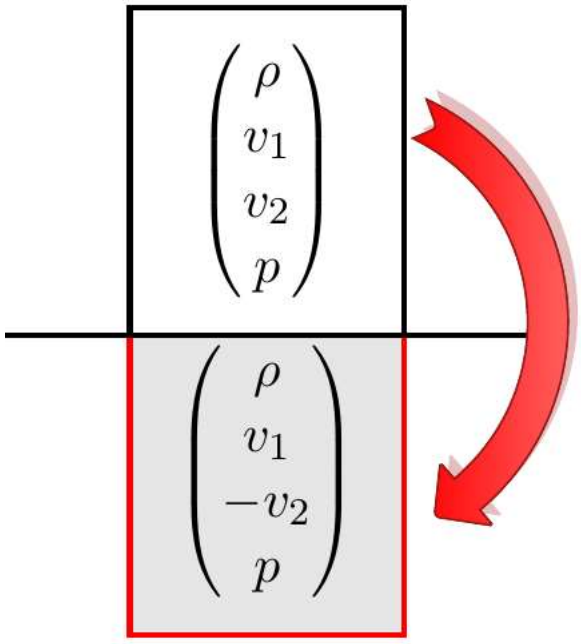
Physical Boundary Conditions

- Walls
- Symmetry
- Periodic
- Freestream
- Inflow-/Outflow (Super-/subsonic)

Boundary Conditions: Euler or Slip Wall

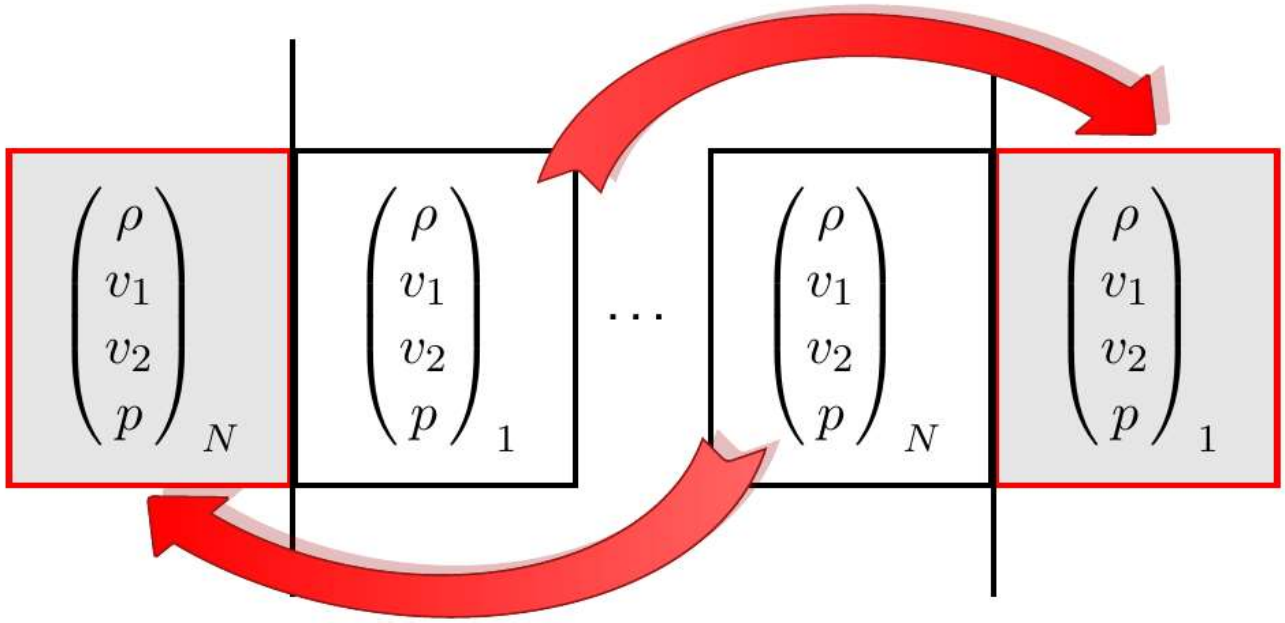
- This boundary condition acts as a symmetry condition.
- Symmetric problems can be cut at symmetry planes, reducing the domain and saving computational time.
- Euler eq. are inviscid; heat flux is zero



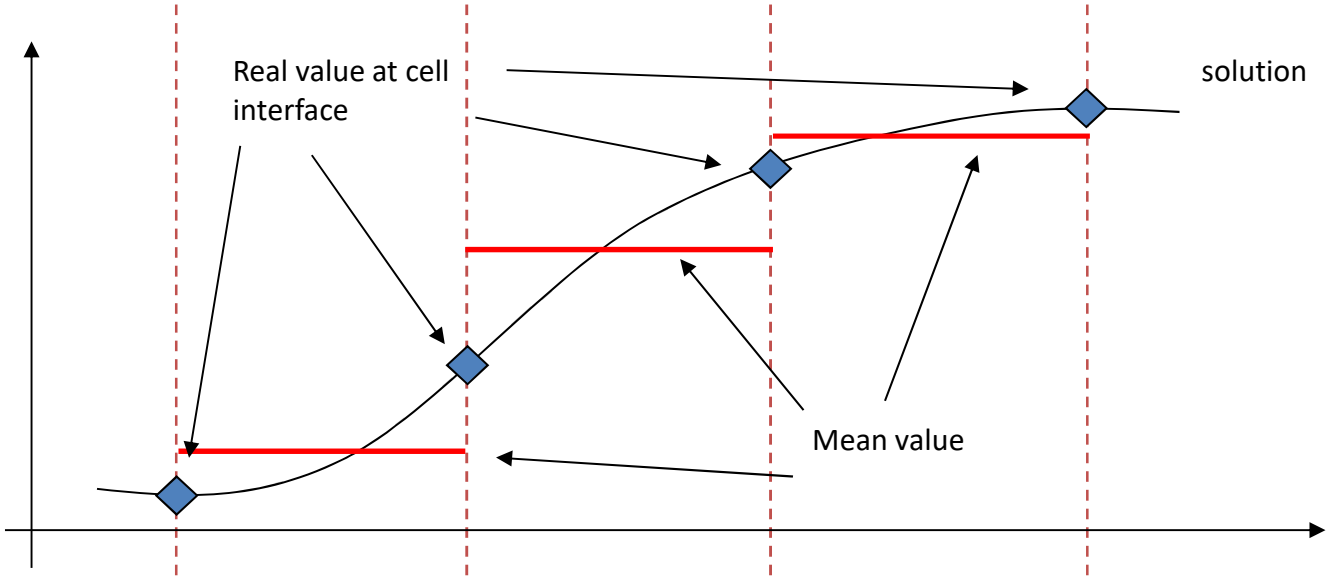


Boundary Conditions: Periodic Sides

- Infinite domain, e.g., channel flow

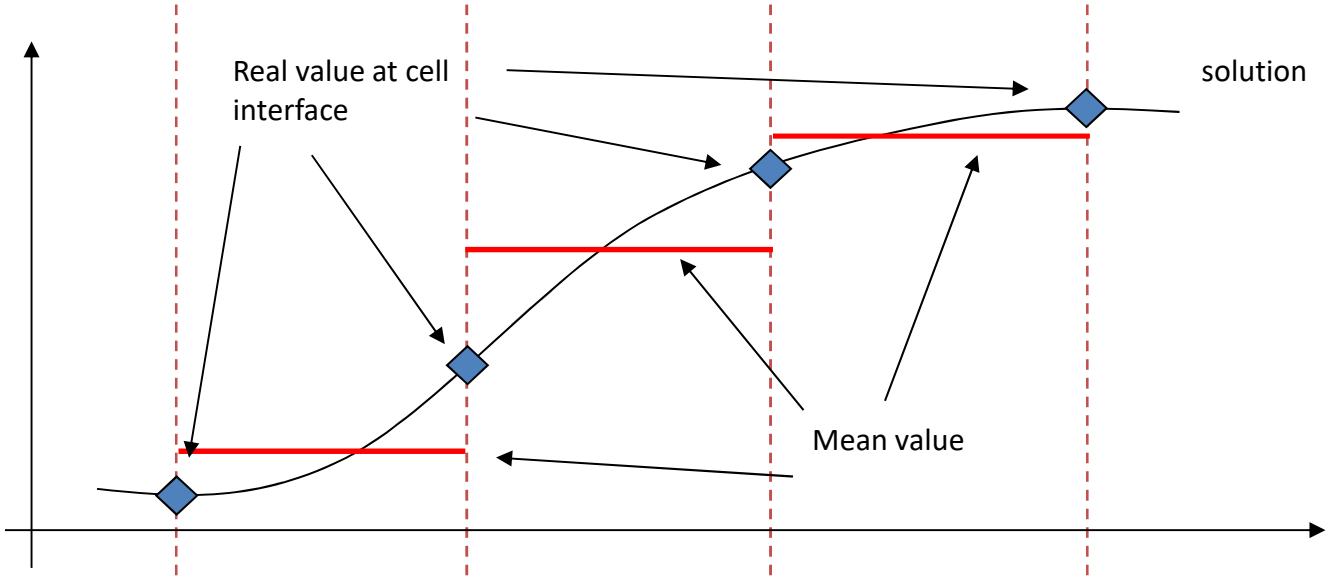


Outlook: Is first-order enough?



--> Problem: Values at cell interfaces differ from the real solution.

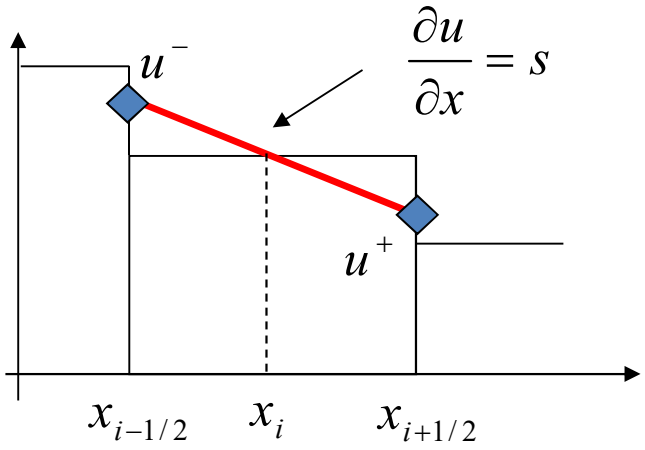
FV-Discretization



Problem: Values at cell interfaces differ from the real solution.
➞ 1st order

Reconstruction in 1D (MUSCL)

Ansatz: Instead of a constant approximation in each cell, a linear distribution is used. The integral value must be preserved.



$$u(x) = \overline{u}_i + s(x - x_i)$$

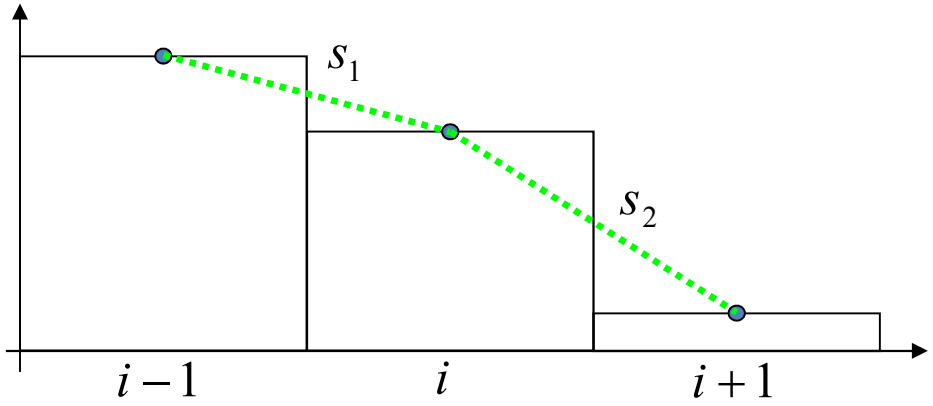
$$u_i^+ = \overline{u}_i + s\frac{\Delta x}{2}$$

$$u_i^- = \overline{u}_i - s\frac{\Delta x}{2}$$

Problem: The FV method has no possibility to save interior cell information beside the mean value.

Reconstruction in Space: Slope Calculation

Process: Only cell mean values are saved. The slope in each cell is calculated by using adjoined cells. Two neighbors allow the computation of two gradients (s_1,s_2).



$$s_1 = \frac{u_i - u_{i-1}}{\Delta x}$$

$$s_2 = \frac{u_{i+1} - u_i}{\Delta x}$$

Problem: Which one is the correct gradient to reconstruct the cell value?

Reconstruction: TVD

TVD-Property (Total Variation Diminishing)

$$\sum_{all\ i} \left| u_{i+1}^{n} - u_{i}^{n} \right| \leq \sum_{all\ i} \left| u_{i+1}^{0} - u_{i}^{0} \right|$$

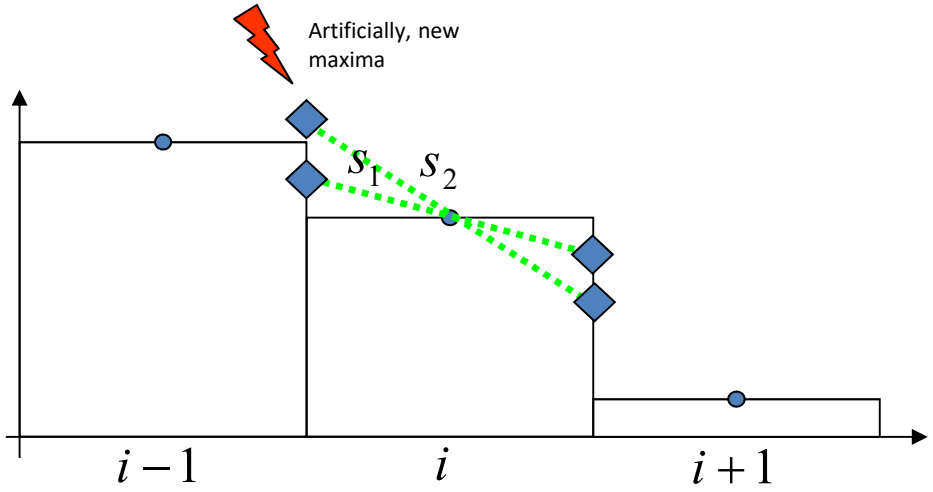
Sufficiency (A. Harten)

$$0 \leq \left\{ \frac{\Delta x\, s_i}{u_i - u_{i-1}}, \frac{\Delta x\, s_i}{u_{i+1} - u_i} \right\} \leq 2$$

Interpretation of TVD

TVD:

-Limitation of gradients, no new maxima or minima



The reconstructed slope has to be limited. → Limiter

Reconstruction in 1D: Limiter

1. Minmod-function

$$s_i = \frac{1}{\Delta x} \ \text{minmod} \ \left(u_{i+1} - u_i \ , u_i - u_{i-1}\right)$$

$$\text{minmod} \quad \left(a \ , b\right) = \begin{cases} a & if \quad |a| < |b| \ , ab > 0 \\ b & if \quad |a| \geq |b| \ , ab > 0 \\ 0 & else \end{cases}$$

2. Sweby‘s slope calculation

$$s_k\left(a \ , b\right) = sign(a) \ \max \ \left\{\left|\text{minmod} \ \left(a \ , kb\right)\right| , \left|\text{minmod} \ \left(ka \ , b\right)\right|\right\}$$
$$mit \quad 1 \leq k \leq 2$$

Reconstruction in 2 and 3 Dimensions

Cartesian Grid:

Each dimension is independent of the other. The 1D-scheme can be applied for each dimension.

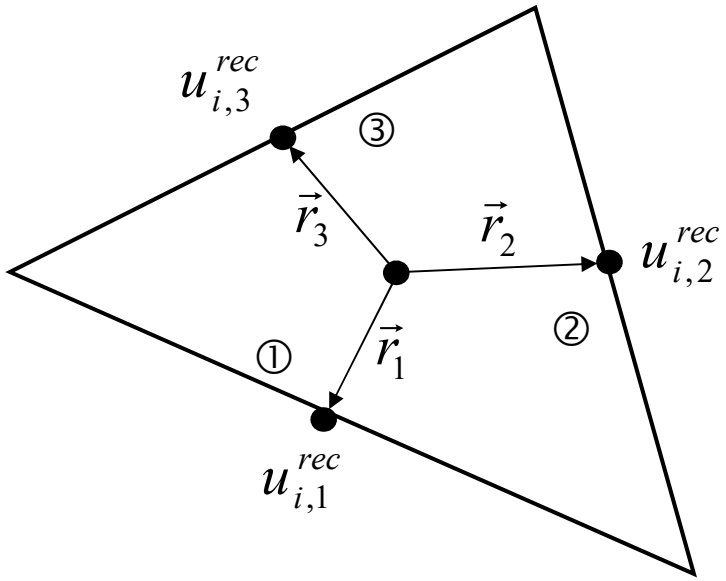
Unstructured Grid:

More complex. Dimensions cannot be separated. More complex slope and limiter calculation.

Reconstruction: Unstructured Grid

Barth & Jespersen:

$$u_{i.m}^{rec} = u_i + \nabla u_i \cdot \vec{r}_m$$



Limiter is necessary!

$$u_{i.m}^{rec} = u_i + \psi_i \nabla u_i \cdot \vec{r}_m$$

Methods of Second Order in Time

Two different methods are possible:

1. „Method of Lines“
   - Separation of space and time integration
   - Time integration can easily exchanged
   - Time and space order independent
   - Easy to implement

2. „Space-Time-Expansion“
   - Time order equals space order
   - Depends on equation system
   - Difficult to implement

Method of Lines

The time discretization is independent of the spatial discretization.

$$\int_V \frac{\partial}{\partial t} u(x,t)\, dV + \int_V \nabla \cdot f\left(u(x,t)\right) dV = 0$$

$$\Rightarrow \frac{\partial \overline{u}(t)}{\partial t} = -\frac{1}{V} \oint_{\partial V} f\left(u(x,t)\right) \cdot \vec{n}\; dV$$

Definition of spatial operator

$$R = -\frac{1}{V} \oint_{\partial V} f\left(u(x,t)\right) \cdot \vec{n}\; dV$$

Results in an ordinary differential equation

$$\frac{\partial \overline{u}(t)}{\partial t} = R$$

Method of Lines - Implementation

1. Spatial operator R is computed with arbitrary numerical scheme

$$R = L(u,t)$$

2. Solving the ordinary differential equation (ODE) by a method for initial value problems. Typical methods are explicit Runge-Kutta methods or implicit BDF methods. A second order method is the improved Euler-scheme.

$$k_1 = L(u_n, t_n)$$

$$k_2 = L\left(u_n + \frac{\Delta t}{2} k_1, t_n + \frac{\Delta t}{2}\right)$$

$$u_{n+1} = u_n + \Delta t k_2$$

Space-Time-Expansion

Solve following equation:

$$\int_V \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial t} u(x,t)\, dt\, dV + \int_{t_n}^{t_{n+1}} \int_V \nabla \cdot f\left(u(x,t)\right) dV\, dt = 0$$

$$\Rightarrow u_i^{n+1} = u_i^{n+1} - \frac{1}{V} \int_{t^n}^{t^{n+1}} \sum_{m=1}^{N} g\left(u_L^{(m)}(t), u_R^{(m)}(t)\right) \vec{n}\ A_m\ dt$$

First order methods use the rectangle method. For second order, a more accurate integration method has to be used, e.g. the midpoint method.

$$\int_{t^n}^{t^{n+1}} \sum_{m=1}^{N} g\left(u_L^{(m)}(t), u_R^{(m)}(t)\right) \vec{n}\ A_m\ dt = \Delta t \sum_{m=1}^{N} g\left(u_L^{(m)}\left(t^{n+1/2}\right), u_R^{(m)}\left(t^{n+1/2}\right)\right) \vec{n}\ A_m$$

The states $u_{L,R}^{(m)}\left(t^{n+1/2}\right)$ are unknowns.

Space-Time-Expansion: Cauchy-Kovalevskaya-Procedure

The unknown state $u_{L,R}^{(m)}\left(t^{n+1/2}\right)$ is approximated by a Taylor series

$$u\left(t^{n+1/2}\right)=u\left(t^{n}\right)+\frac{\Delta t}{2}u_t\left(t^{n}\right)+O\left(\Delta t^2\right)$$

The cell local differential equation is:

$$u_t+\nabla\cdot f\left(u\right)=0$$

The idea of the Cauchy-Kovalevskaya procedure is to express the time derivative by the differential equation.

$$u_t=-\nabla\cdot f\left(u\right)$$

Space-Time-Expansion

The unknown state at the cell midpoints is expressed by a Taylor series in space and time:

$$
\begin{aligned}
u\left(x, y, z, t^{n+1/2}\right) \quad = \quad & u\left(x_0, y_0, z_0, t^n\right) + \\
& \Delta x\, u_x\left(x_0, y_0, z_0, t^n\right) + \\
& \Delta y\, u_y\left(x_0, y_0, z_0, t^n\right) + \\
& \Delta z\, u_z\left(x_0, y_0, z_0, t^n\right) + \\
& \frac{\Delta t}{2} u_t\left(x_0, y_0, z_0, t^n\right) + \\
& O\left(\Delta x^2, \Delta y^2, \Delta t^2\right)
\end{aligned}
$$

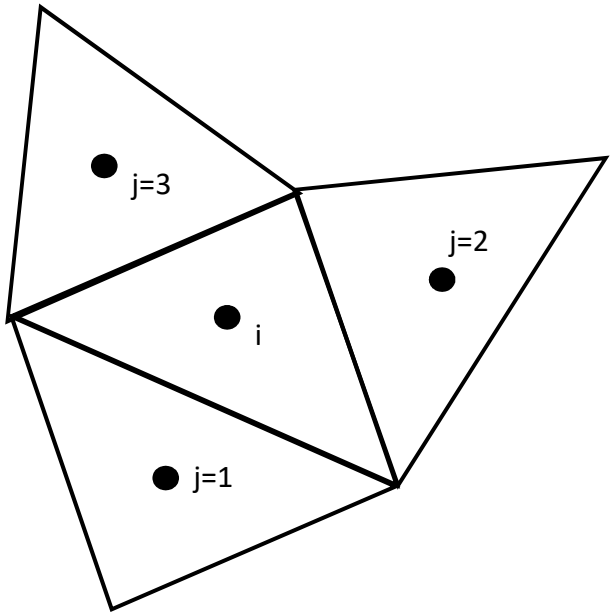
The spatial derivative is approximated by the reconstruction step. The time derivative is computed by the Cauchy-Kovalevskaya procedure. This allows to compute the state at each point of a space time element for the flux computation.

Slope Calculation on Unstructured Grids

Least-Squares-Method:

$$M^T M \nabla u_i = M^T \tilde{u}$$

$$\tilde{u} = \begin{pmatrix} u_{j=1} - u_i \\ u_{j=2} - u_i \\ \vdots \\ u_{j=N} - u_i \end{pmatrix} \quad M = \begin{pmatrix} \Delta x_{j=1} & \Delta y_{j=1} & \Delta z_{j=1} \\ \Delta x_{j=2} & \Delta y_{j=2} & \Delta z_{j=2} \\ \vdots & \vdots & \vdots \\ \Delta x_{j=N} & \Delta y_{j=N} & \Delta z_{j=N} \end{pmatrix}$$



$$\boxed{\nabla u_i = G_i \tilde{u}} \quad \text{mit} \quad G_i = \left(M^T M\right)^{-1} M^T$$

The matrix G_i can be computed for each cell in a preprocessing step. The reconstruction simplifies to a matrix-vector operation per time step.

Limiter on Unstructured Grids

Barth und Jespersen:

Requires:

1. Maxima of all four cells:

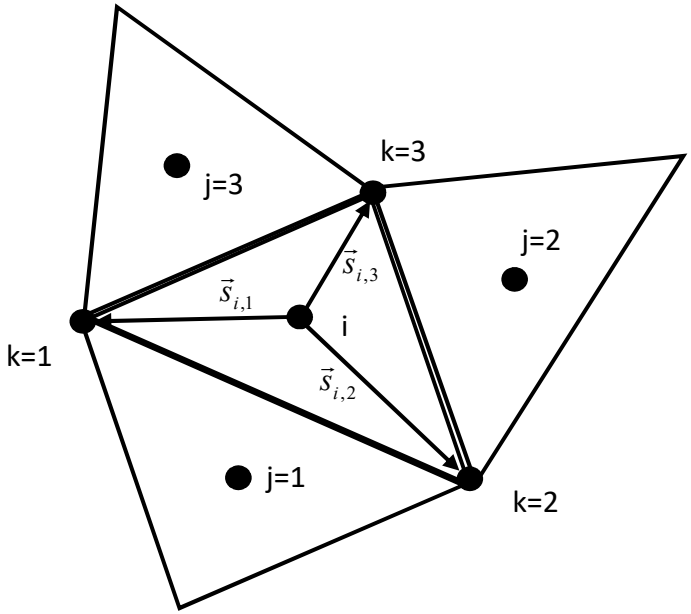
$$u_{\max} = \max\left(u_i, u_{j=1}, \ldots, u_{j=N}\right)$$

2. Minima of all four cells:

$$u_{\min} = \min\left(u_i, u_{j=1}, \ldots, u_{j=N}\right)$$

3. Vectors from barycenter to each node:

$$\vec{s}_{i,k}, \quad k = 1 \ldots N$$



Limiter

Computing the limitation ψ_k for each node:

$$\psi_k = \begin{cases} \min\left(1, \dfrac{u_{\max} - u_i}{\nabla u_i \cdot \vec{s}_{i,k}}\right) & \text{if } \nabla u_i \cdot \vec{s}_{i,k} > 0 \\ \min\left(1, \dfrac{u_{\min} - u_i}{\nabla u_i \cdot \vec{s}_{i,k}}\right) & \text{if } \nabla u_i \cdot \vec{s}_{i,k} < 0 \\ 1 & \text{if } \nabla u_i \cdot \vec{s}_{i,k} = 0 \end{cases}$$

The limited slope is

$$\nabla u_i^{(\lim)} = \min_k(\psi_k) \nabla u_i$$