

Incompressible Flows

Dr. Albert Ruprecht

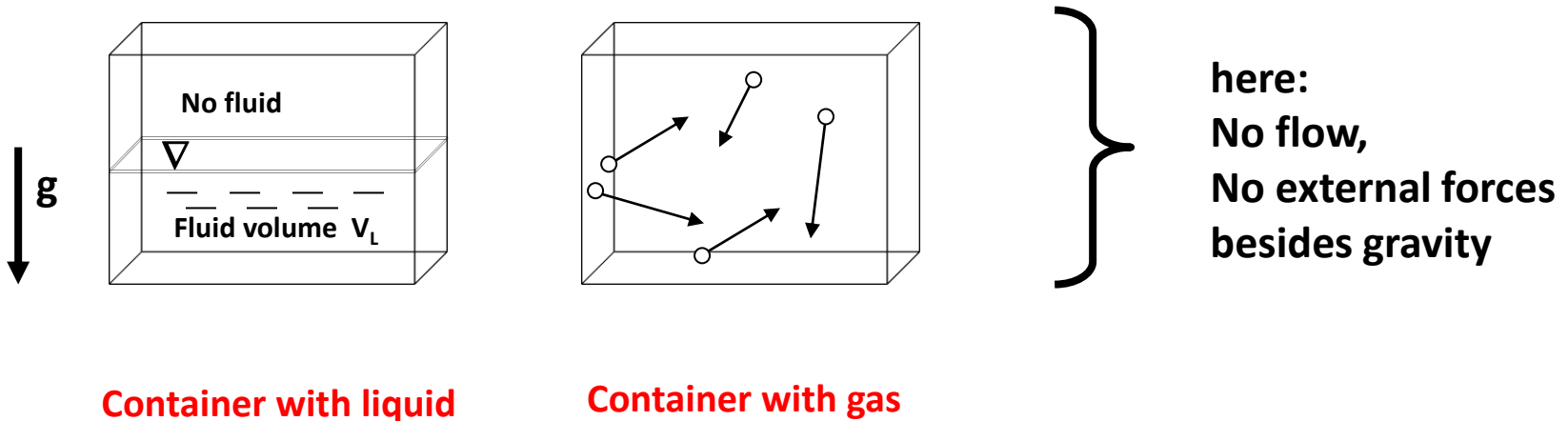
Former: Institute of Fluid Mechanics
and Hydraulic Machinery

University of Stuttgart

Fluids and gases

Fluid (flowing medium): gas or liquid

- Solid body** : dedicated volume, dedicated shape
- Liquid** : dedicated volume, no dedicated shape
- Gas (Steam)** : no dedicated volume, no dedicated shape



Fluids and gases

Container with liquid

$$V_{Fluid} \leq V_{container}$$

$$\rho = \frac{M_{Fluid}}{V_{Fluid}}$$

Constant density

Container with gas

$$V_{Fluid} = V_{container}$$

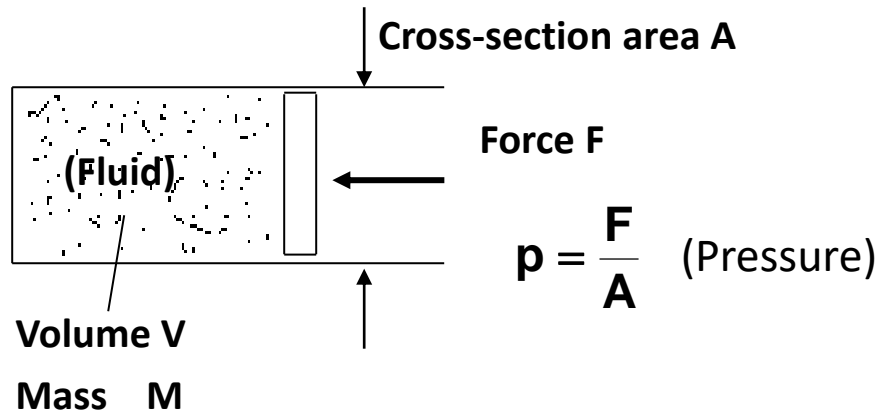
$$\rho = \frac{M_{Fluid}}{V_{container}}$$

variable density

V	m ³	Volume
M	kg	Mass
ρ	$\frac{kg}{m^3}$	Density
$\left(v = \frac{1}{\rho} \right)$	$\frac{m^3}{kg}$	specific Volume

Fluids and gases

Compressibility of a fluid (isotherm)



Experiment:

Increasing load

$F \rightarrow F + dF$, $p \rightarrow p + dp$

$V \rightarrow V + dV$ (with $dV < 0$)

$\rho \rightarrow \rho + d\rho$

$$\frac{d\rho}{\rho} = \gamma_T \frac{dp}{p} \Leftrightarrow \frac{p}{\rho} = \gamma_T \frac{dp}{d\rho}$$

Isothermal
Compressibility coefficient

Water $\gamma_T = 45,4 \cdot 10^{-6}$

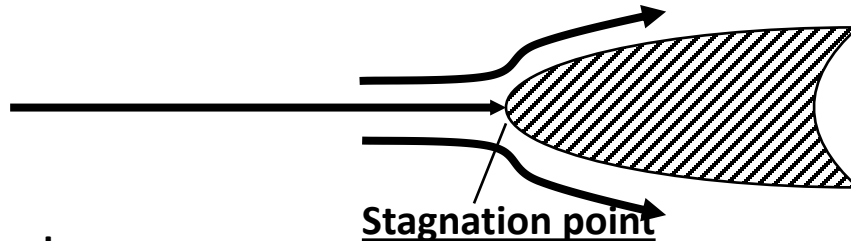
Ideal gas $\gamma_T = 1$

weil für $T = \text{const} \rightarrow \frac{p}{\rho} = RT = \text{const.} = \frac{dp}{d\rho}$

The compressibility of liquids is several orders of magnitude smaller compared to gases

Fluids and gases

Compressibility of a gas flow



Velocity $u \rightarrow 0$
 Pressure $p \rightarrow p + dp$
 Density $\rho \rightarrow \rho + d\rho$

Example:
 Stagnation flow

Stagnation point
 Velocity is zero
 Pressure increases
 Density can increase

incompressible: Density is constant, only pressure increases
compressible: Pressure change is large enough, that it results in a significant change of density (compression)

Bernoulli equation: $p + \frac{1}{2} \rho u^2 = \text{const.}$

$$d\left(p + \frac{1}{2} \rho u^2\right) = 0$$

$$dp + \frac{1}{2} u^2 d\rho + \frac{1}{2} \rho d(u^2) = 0$$

$$\left. \frac{d\rho}{\rho} = -\frac{1}{2} \frac{d(u^2)}{\frac{dp}{d\rho} + \frac{u^2}{2}} \right\}$$

Gas flow can be supposed to be incompressible, when this expression is small

Fluids and gases

Compressibility of a gas flow

$$\frac{dp}{d\rho} = a^2 \quad a: \text{speed of sound}$$

$$\frac{dp}{\rho} = -\frac{1}{2} \frac{d(u^2)}{a^2 + \frac{u^2}{2}} = -\frac{1}{2} \frac{d\left(\frac{u^2}{a^2}\right)}{1 + \frac{1}{2} \frac{u^2}{a^2}} = -\frac{1}{2} \frac{d(\text{Ma}^2)}{1 + \frac{1}{2} \text{Ma}^2}$$

This expression is small, when the Mach number is small, approximately **Ma < 0.3**

$$\text{Ma} = \frac{u}{a} \quad \text{Mach number}$$

$$a = \sqrt{\kappa \cdot R \cdot T} \quad \text{Ideal gas}$$

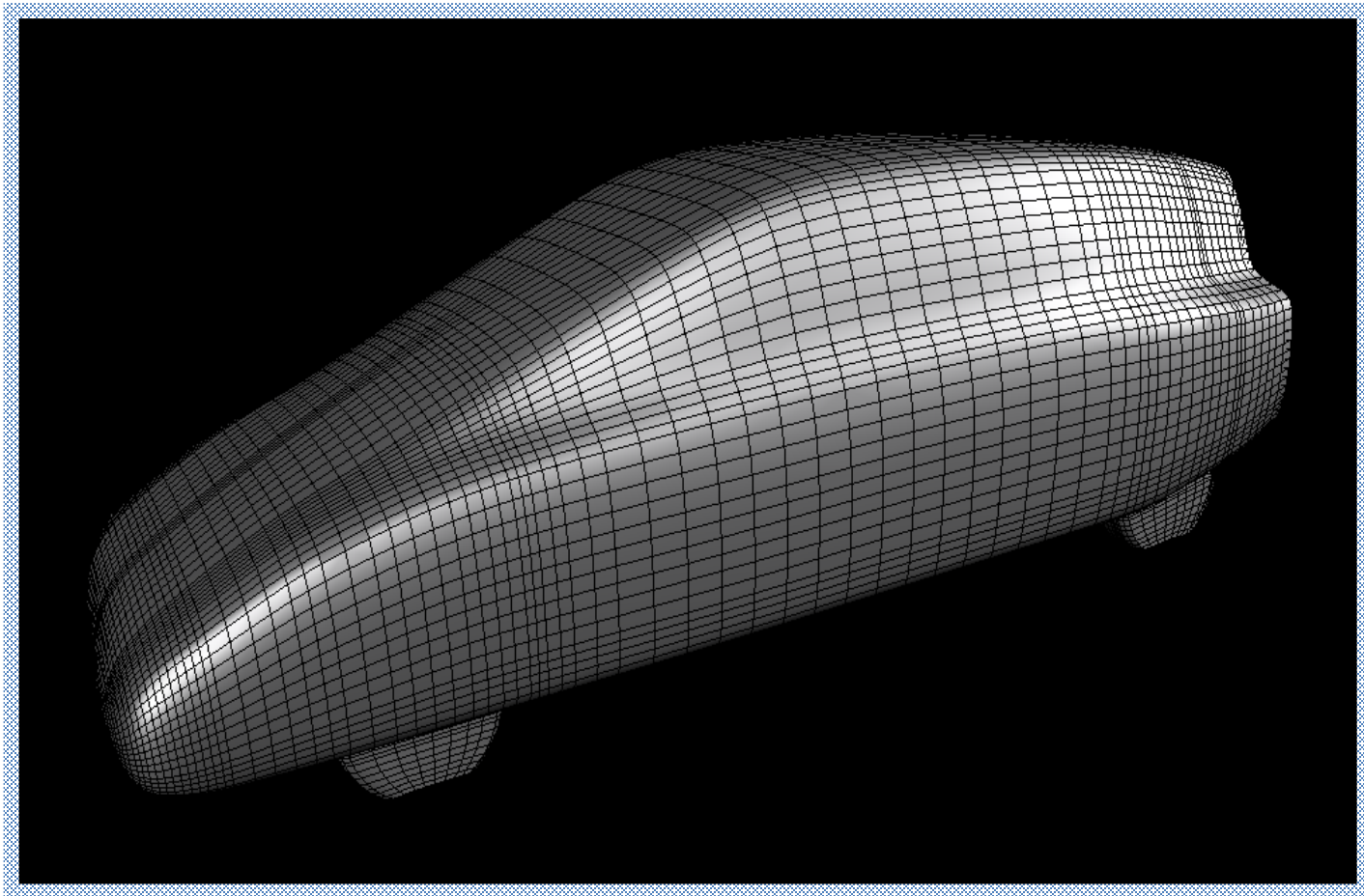
Examples:

	Mach number
Ventilation systems	0.1
Flow around cars	0.2-0.3
Wind rotor	0.2-0.5

Steam turbine	0.7
Flow around Airplane	0.9
Explosion	>1,5

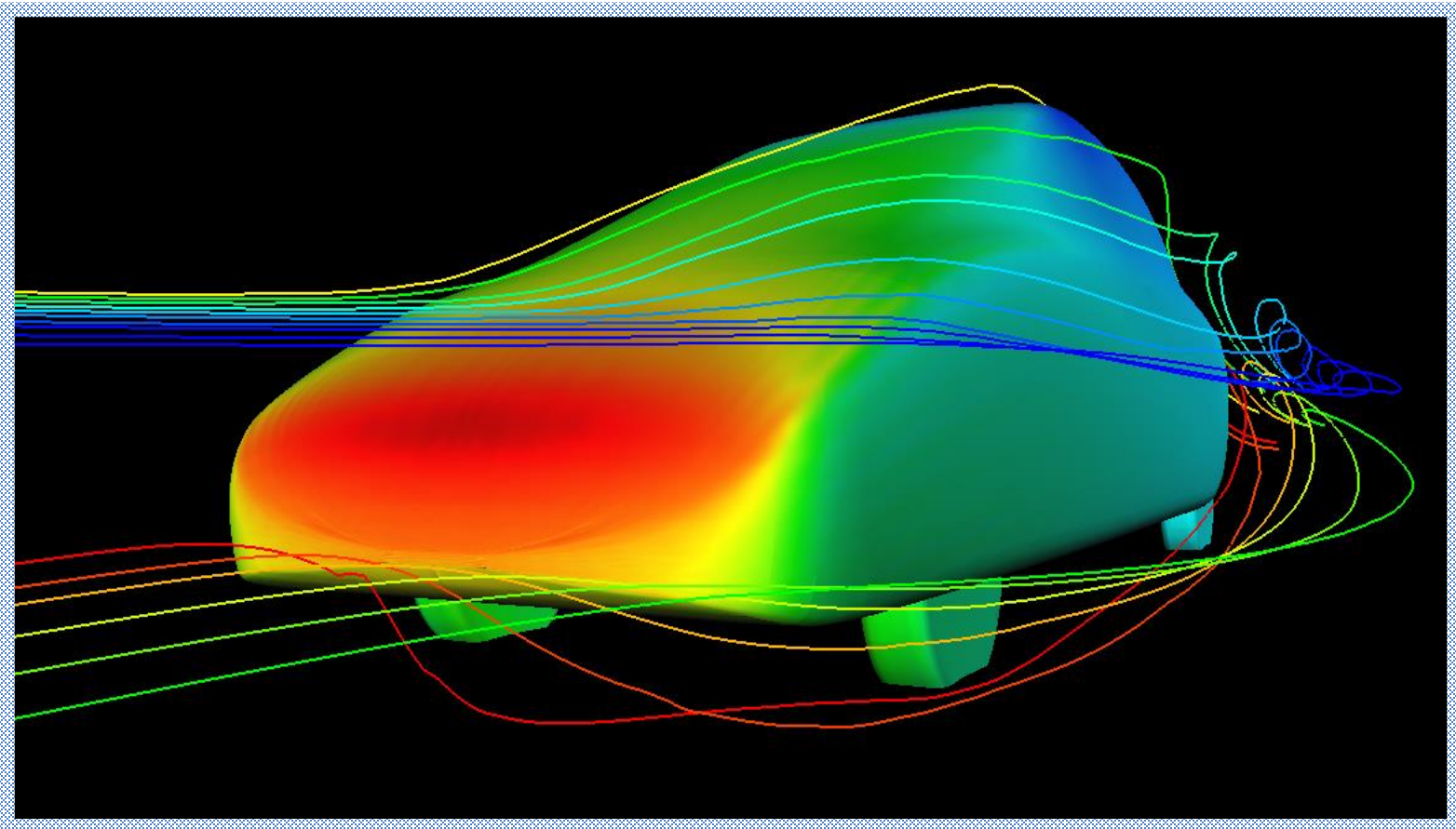
Example: Incompressible Fluid

Flow around a car



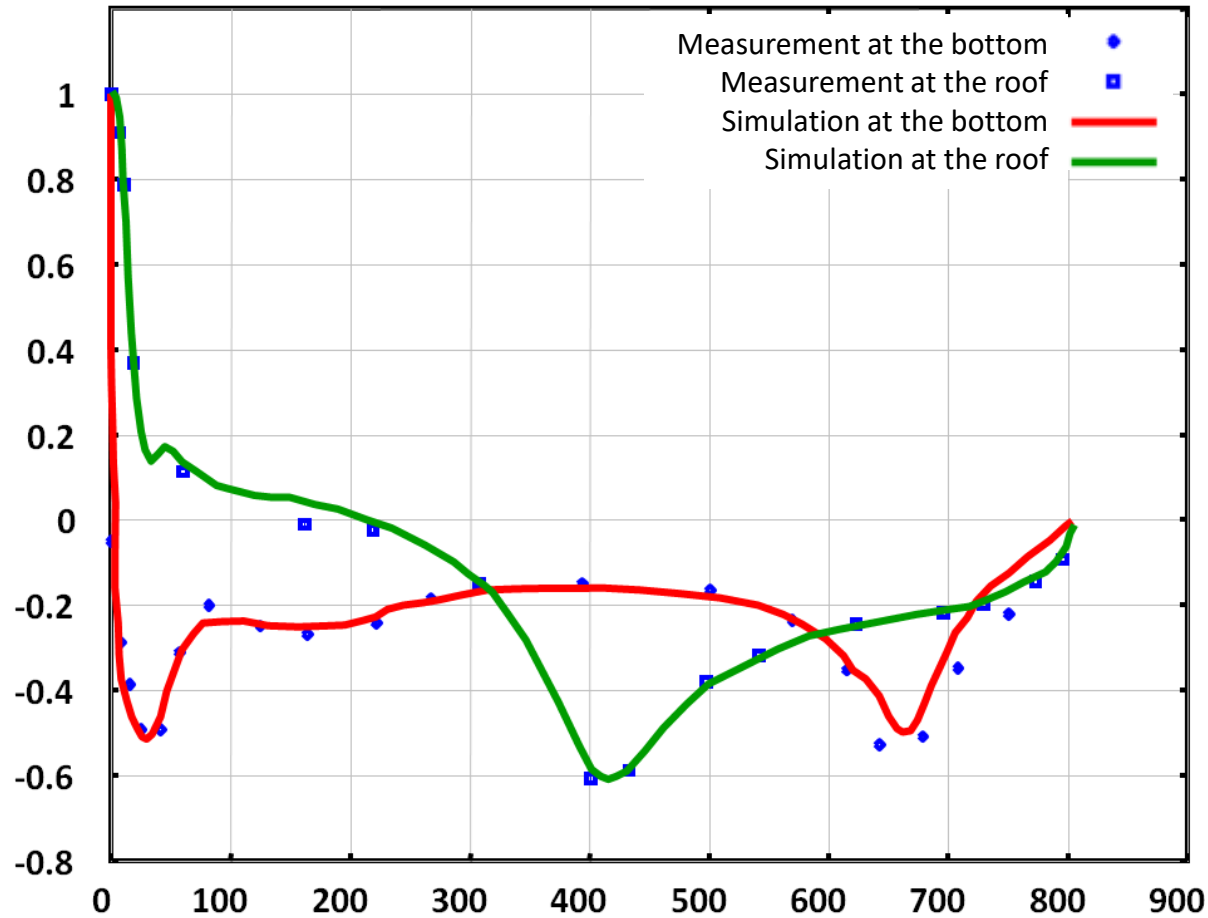
Example: Incompressible Fluid

Flow around a car



Example: Incompressible Fluid

Flow around a car



Pressure distribution in the middle of the car

Compressible Fluids

Speed of sound $a_{FI} = \sqrt{\frac{E_{FI}}{\rho}}$

← Elasticity modulus
← Density

1 m³
1 bar pressure increase

	E_{FI} [bar]	ρ [kg/m ³]	a_{FI} [m/s]
Water	$2.06 \cdot 10^4$	1000	1440
Crude oil	$1.56 \cdot 10^4$	900	1310
Petrol	$0.88 \cdot 10^4$	750	1080
Air	$0.13 \cdot 10^1$	1.2	330

 0.9995 m³

 0.23 m³

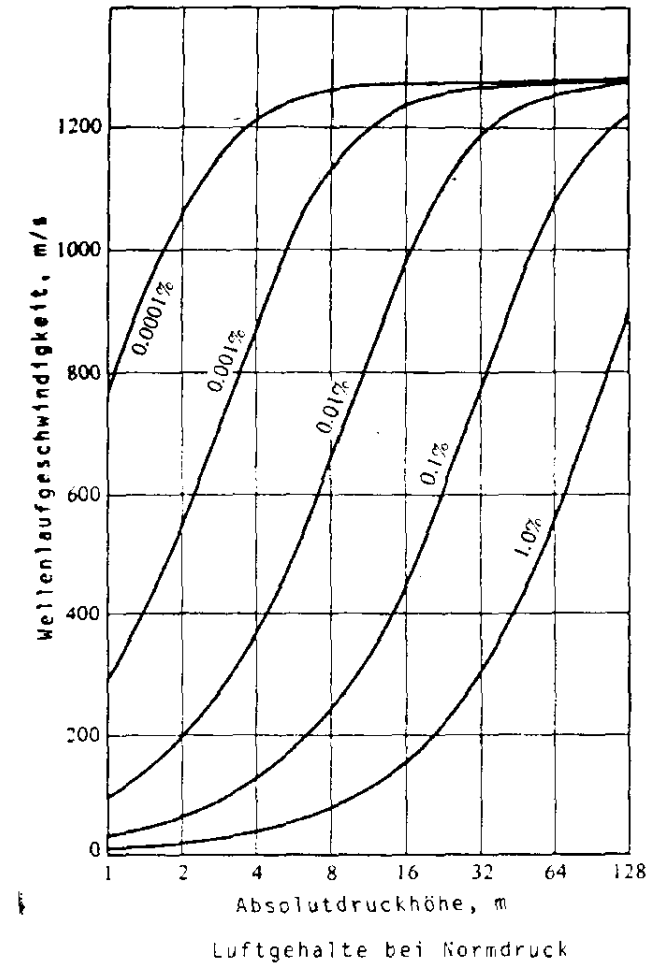
Compressible Fluids

Speed of sound

Water: appr. 1400 m/s

Air: appr. 330 m/s

Speed of sound in water decreases significantly with dissolved air



Compressible Fluids

Example: Compressible pipe flow

Eigenfrequencies:

1st Eigenfrequency: $f_1 = a/(4L)$

2nd Eigenfrequency: $f_2 = 3a/(4L)$

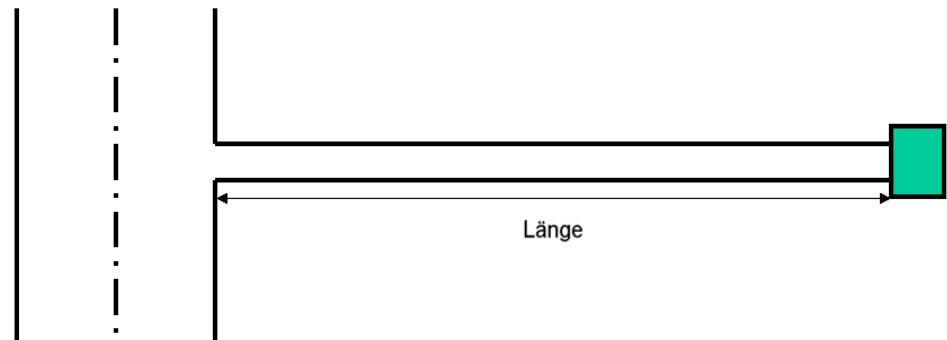
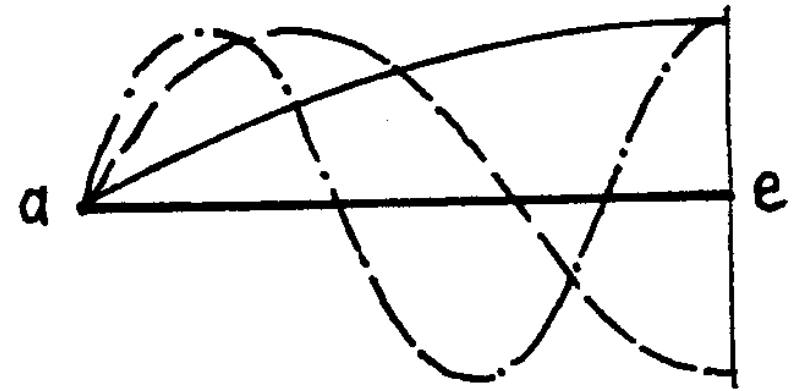
3rd Eigenfrequency: $f_3 = 5a/(4L)$

Example:

Length: 5 m

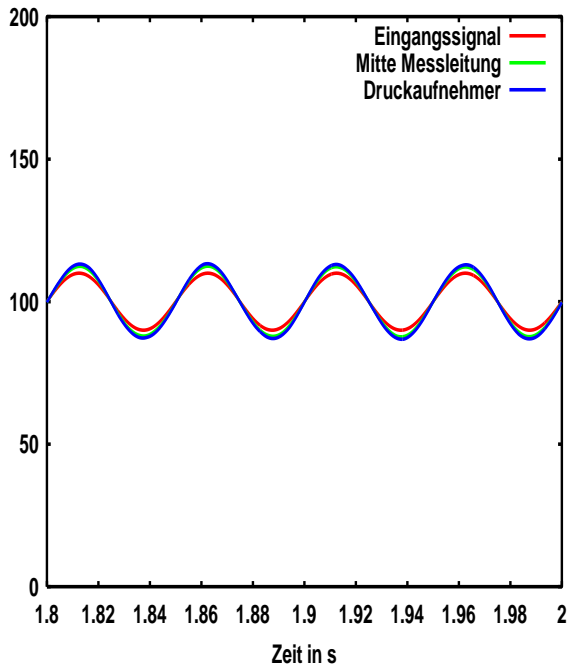
Speed of sound: 900 m/s

First Eigenfrequency: 45 Hz

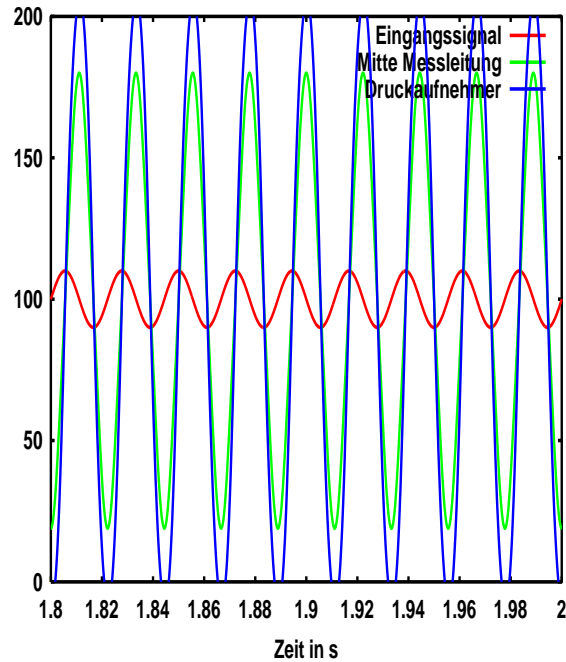


Compressible Fluids

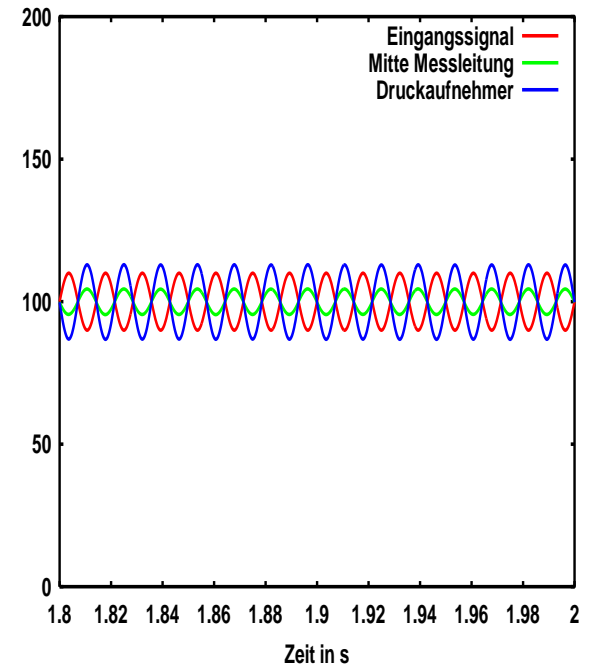
Exciting frequency 20Hz



Exciting frequency 45Hz

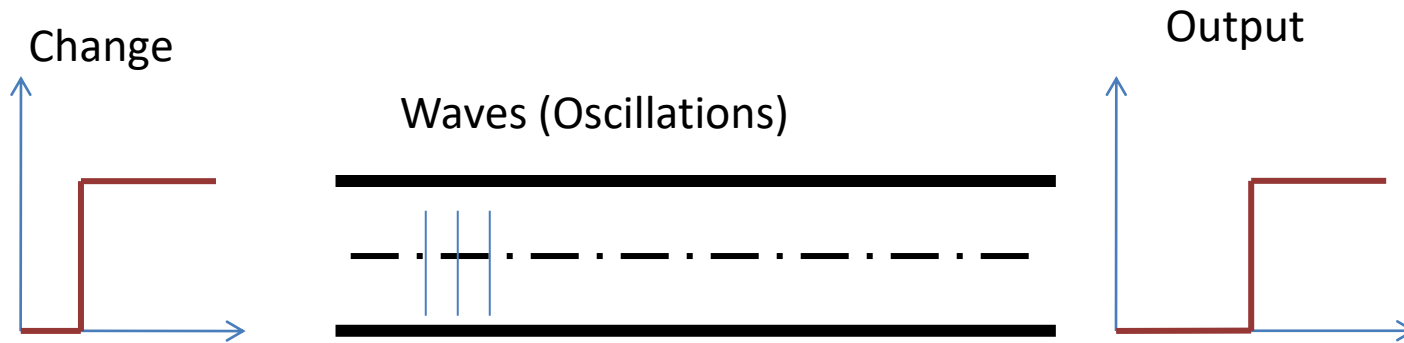


Exciting frequency 70 Hz

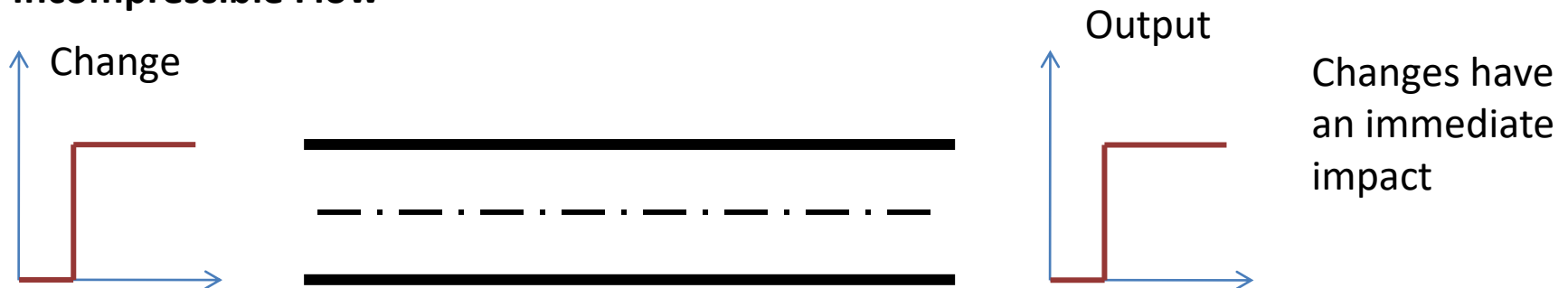


Difference in behavior of compressible and incompressible flows

Compressible Flow



Incompressible Flow



For oscillations and resonance phenomena compressibility can be very significant

Compressible Navier-Stokes equations

Mass conservation:
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0$$

Momentum conservation:

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} - \rho \frac{\partial \Psi}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \frac{\partial u_k}{\partial x_k} \delta_{ij} \right]$$

Equation of state (e. g. ideal gas law):

$$p = \rho R T$$

Energy conservation:
$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} - \alpha \left(\frac{\partial}{\partial x_j} \frac{\partial T}{\partial x_j} \right) = \Phi$$

Incompressible flow

Compressible mass conservation:
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0$$

Incompressible flow: $\rho \neq f(p), \rho \neq f(t)$ Mostly: $\rho = \text{const.}$

This results in

Continuity equation:
$$\frac{\partial u_i}{\partial x_i} = 0$$

Momentum equations:
$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} - \rho \frac{\partial \Psi}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right]$$

Dimensionless equations

The equations were made dimensionless by

- a characteristic length L and
- a characteristic velocity \hat{U}

$$u_i^* = u_i / \hat{U}$$

$$x_i^* = x_i / L$$

$$t^* = t / (L / \hat{U})$$

$$p^* = p / (\rho \hat{U}^2)$$

Dimensionless equations

Introduced into the Navier-Stokes equations results in

$$\frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*} = -\frac{\partial p^*}{\partial x_i^*} + \frac{\partial}{\partial x_j^*} \left[\frac{1}{Re} \left(\frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*} \right) \right]$$

$$\frac{\partial u_i^*}{\partial x_i^*} = 0$$

With the Reynolds number $Re = \frac{\hat{U}L}{\nu}$



The only relevant characteristic number for incompressible flows is the Reynolds number



Problem: No conditional equation for the pressure

Methods for calculation the pressure

- **Artificial compressibility**
Explicit, Implicit
- **Poisson equation for the pressure**
- **Pressure correction methods**
Usawa
SIMPLE, SIMPLEC, SIMPLEST, PISO
- others
Fractional Step Method
etc

Based on

Physical approximation

Physical modelling

Numerical approximation

Methods for calculation the pressure

- **Artificial compressibility**

Explicit, Implicit

- **Poisson equation for the pressure**

- **Pressure correction methods**

Usawa

SIMPLE, SIMPLEC, SIMPLEST, PISO

- **others**

Fractional Step Method

etc

Based on

“Physical” approximation

Physical modelling

Numerical approximation

Artificial compressibility

Continuity equation
 (Compressible)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0$$

Artificial compressibility

$$\frac{\partial p}{\partial \rho} = a_k^2$$

$a_k \dots$ Artificial compressibility
Numerical coefficient

Modified continuity equation

$$\frac{1}{a_k^2} \frac{\partial p}{\partial t} + \frac{\partial u_i}{\partial x_i} = 0$$

Momentum equation

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right]$$

Correct steady state solution:

$$\frac{1}{a_k^2} \frac{\partial p}{\partial t} \rightarrow 0 \quad \Rightarrow \text{exact Solution}$$

Artificial compressibility

For **unsteady problems**

Introduction of a Pseudo time τ

Modified continuity equation

$$\frac{1}{a_k^2} \frac{\partial p}{\partial \tau} + \frac{\partial u_i}{\partial x_i} = 0$$

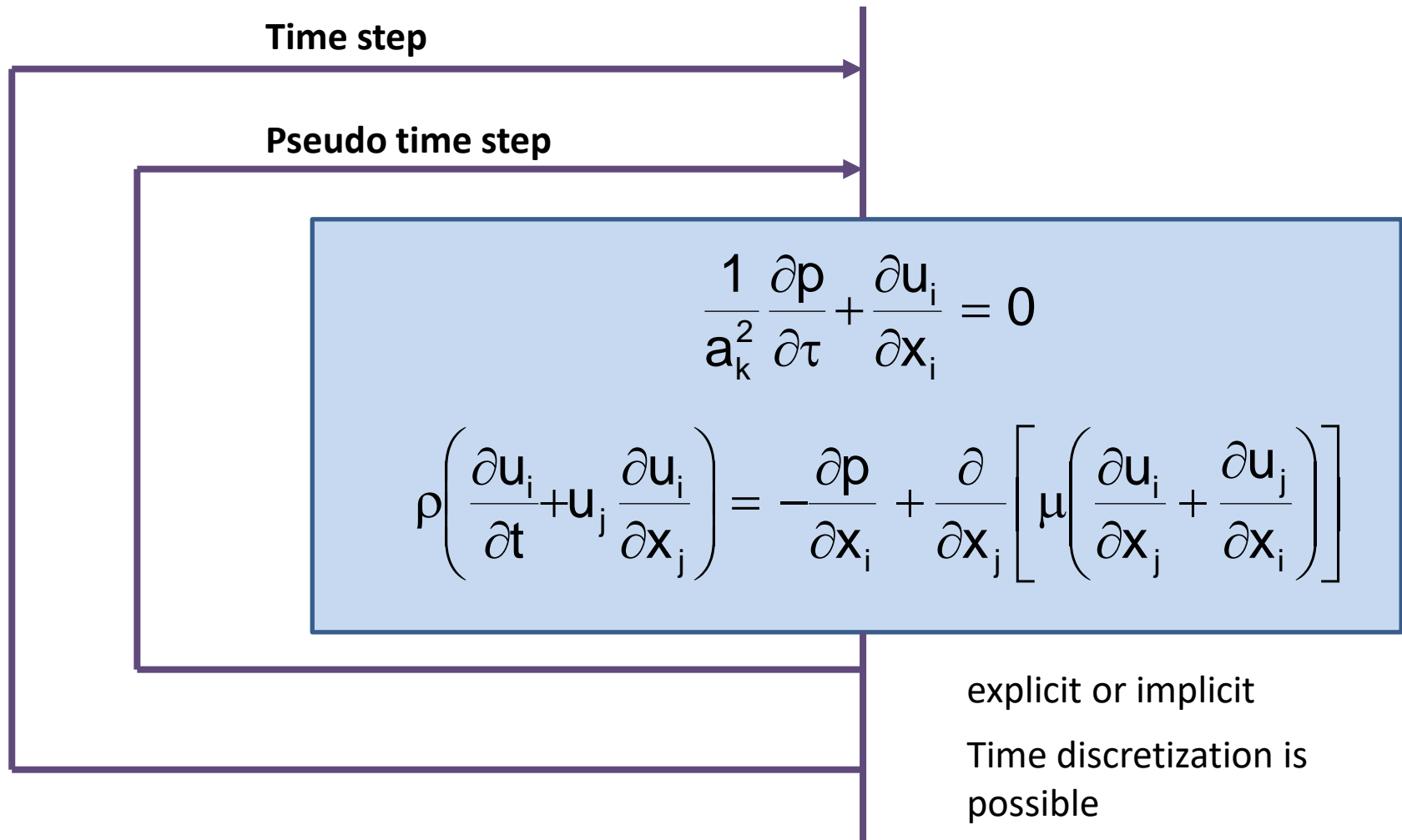
Momentum equations

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right]$$

Integration in each time step

$$\frac{1}{a_k^2} \frac{\partial p}{\partial \tau} \rightarrow 0 \quad \Rightarrow \text{Exact Solution}$$

Artificial compressibility



Methods for calculation the pressure

- **Artificial compressibility**

Explicit, Implicit

- **Poisson equation for the pressure**

- **Pressure correction methods**

Usawa

SIMPLE, SIMPLEC, SIMPLEST, PISO

- others

Fractional Step Method

etc

Based on

Physical approximation

Physical modelling

Numerical approximation

Poisson equation for the pressure

Taking the momentum equation and derive it

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right]$$

Differentiate the x – equation with respect to x,

Differentiate the y – equation with respect to y

Differentiate the z – equation with respect to z

Sum up the resulting equations leads to

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_i} = - \frac{\partial}{\partial x_i} \left(u_j \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_j} \left[\nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] \right)$$

Exact
 equation for
 the pressure

Poisson equation for the pressure

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_i} = - \frac{\partial}{\partial x_i} \left(u_j \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_j} \left[v \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] \right)$$

With the continuity equation $\frac{\partial u_i}{\partial x_i} = 0$

One obtains the conditional equation for the pressure

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_i} = - \frac{\partial}{\partial x_i} \left(u_j \frac{\partial u_i}{\partial x_j} \right)$$

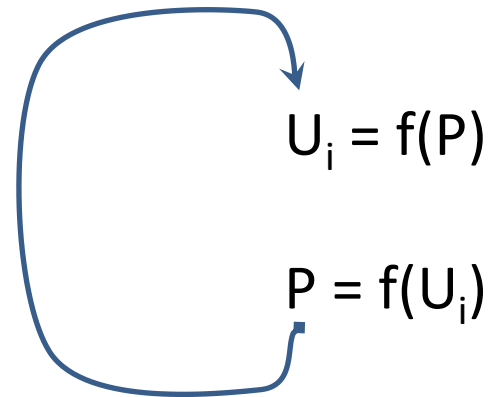
Needed: 2nd order derivatives of
 velocity

Poisson equation for the pressure

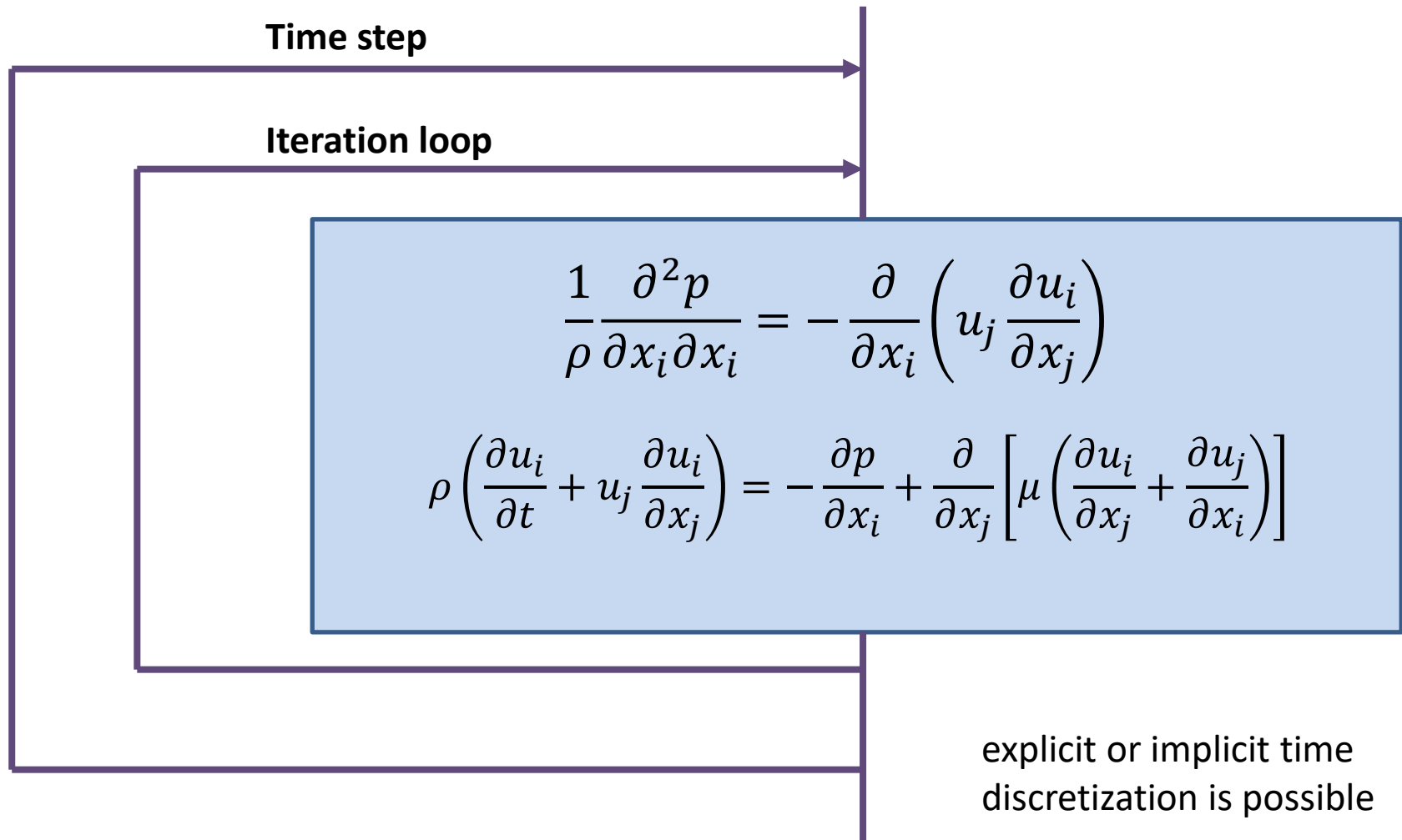
Coupled system, solved iteratively

Velocity is a function of pressure

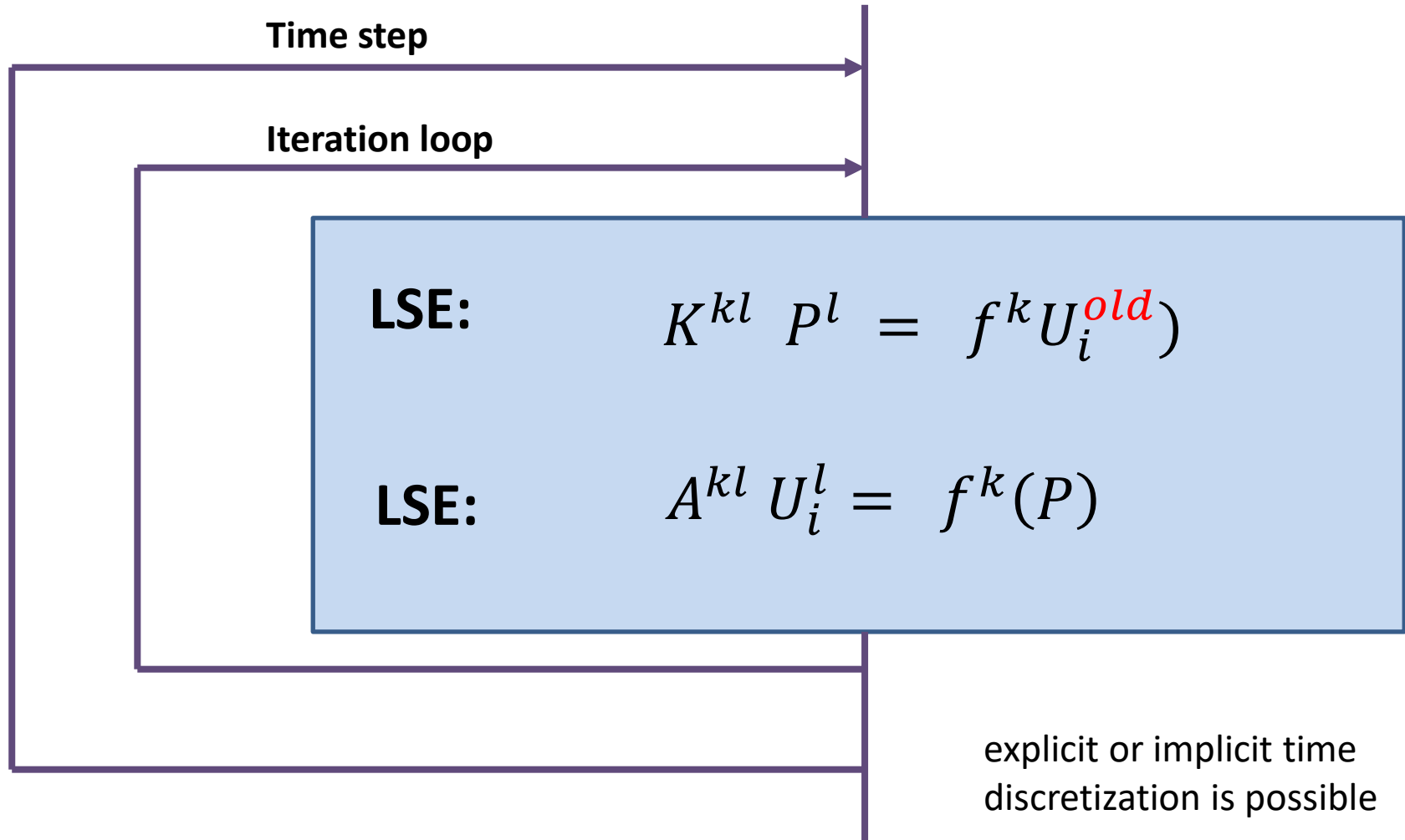
Pressure is a function of velocity



Poisson equation for the pressure



Poisson equation for the pressure



Methods for calculation the pressure

- **Artificial compressibility**
Explicit, Implicit
- **Poisson equation for the pressure**
- **Pressure correction methods**
Usawa
SIMPLE, SIMPLEC, SIMPLEST, PISO
- others
Fractional Step Method
etc

Based on

Physical approximation

Physical modelling

Numerical approximation

Pressure correction methods

Continuity equation: $\frac{\partial u_i}{\partial x_i} = 0$ **Constrain**

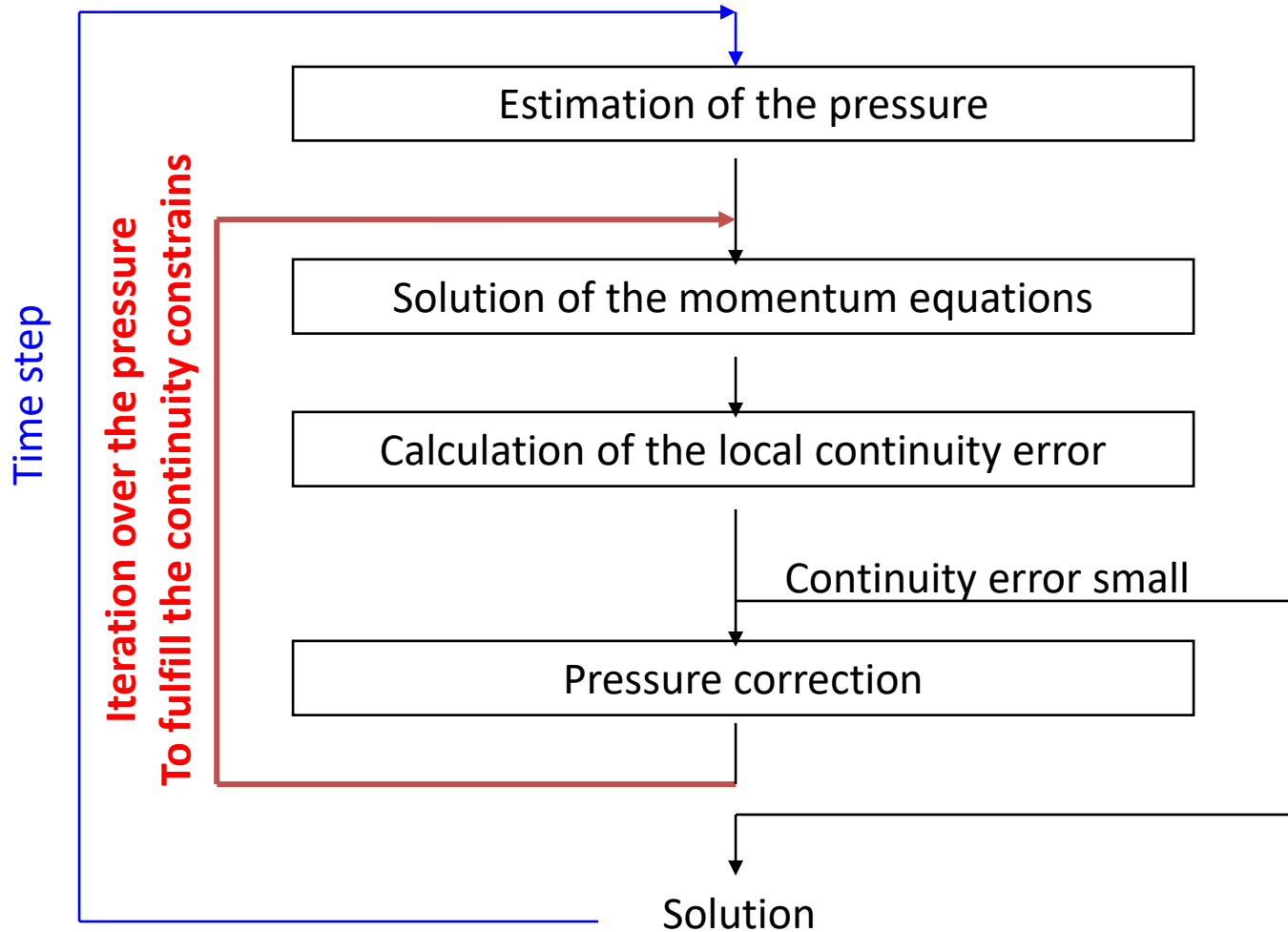
Momentum equation:
$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} - \rho \frac{\partial \Psi}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right]$$

Conditional equation for the velocities

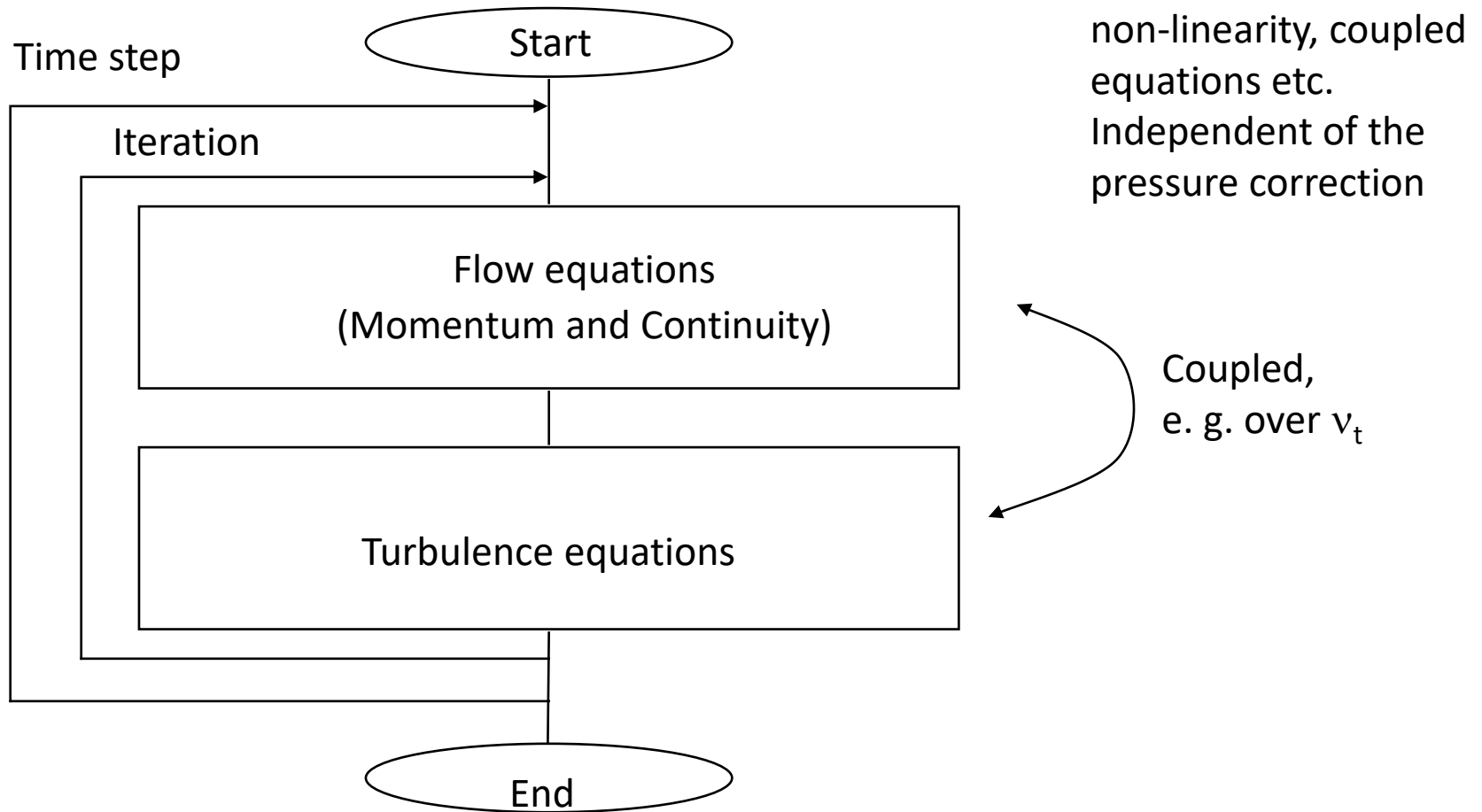
Pressure represents a parameter to fulfill the continuity equation

Mathematical: Pressure is a Lagrange Multiplier

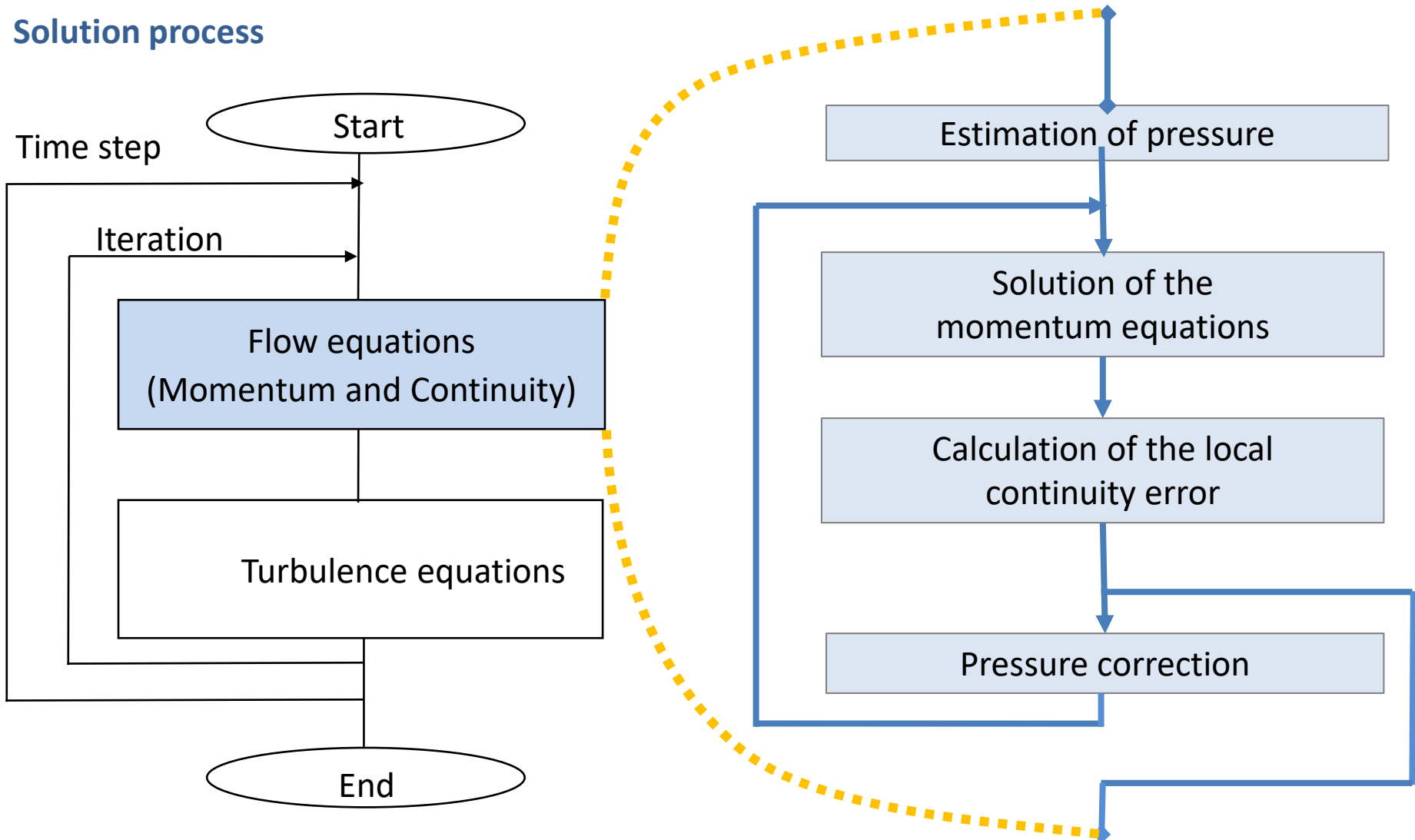
Pressure correction methods



Global iteration procedure



Solution process



Uzawa algorithm

Discretized equations

$$\begin{bmatrix} \underline{\underline{A}} & \underline{\underline{H}} \\ \underline{\underline{Q}} & 0 \end{bmatrix} \begin{bmatrix} \underline{U} \\ \underline{P} \end{bmatrix} = \begin{bmatrix} \underline{b} \\ 0 \end{bmatrix}$$

Momentum equations

Continuity equation

Calculation of the velocities from the momentum equation (1. row)

$$\underline{U} = \underline{\underline{A}}^{-1} \underline{b} - \underline{\underline{A}}^{-1} \underline{\underline{H}} \underline{P}$$

Introduced into the continuity equation (2. row) results in an equation for the pressure

$$\left(\underline{\underline{Q}} \underline{\underline{A}}^{-1} \underline{\underline{H}} \right) \underline{P} = \underline{\underline{Q}} \underline{\underline{A}}^{-1} \underline{b}$$

This equation is very complex and must be solved iteratively.

Uzawa Algorithmus

Pre-conditioned Richardson iteration:

Assumption: \underline{P}^0

Calculation: $\underline{P}^{n+1} = \underline{P}^n - \underline{\rho} \underline{r}^n$ with $\underline{r}^n = \underline{Q} \underline{U}^n$

↑ Conditioning matrix
 ↑ Local continuity error

Acceleration of the convergence by penalization

Uzawa algorithm

Iteration:

Assumption:

$$\underline{P}^0$$

LSE to determine velocity:

$$\left[\underline{A} + \lambda \underline{H} \underline{Q} \right] \underline{U}^n = \underline{b} - \underline{H} \underline{P}^n$$

„Penalty“-parameter

Local continuity error:

$$\underline{r}^n = \underline{Q} \underline{U}^n$$

New pressure:

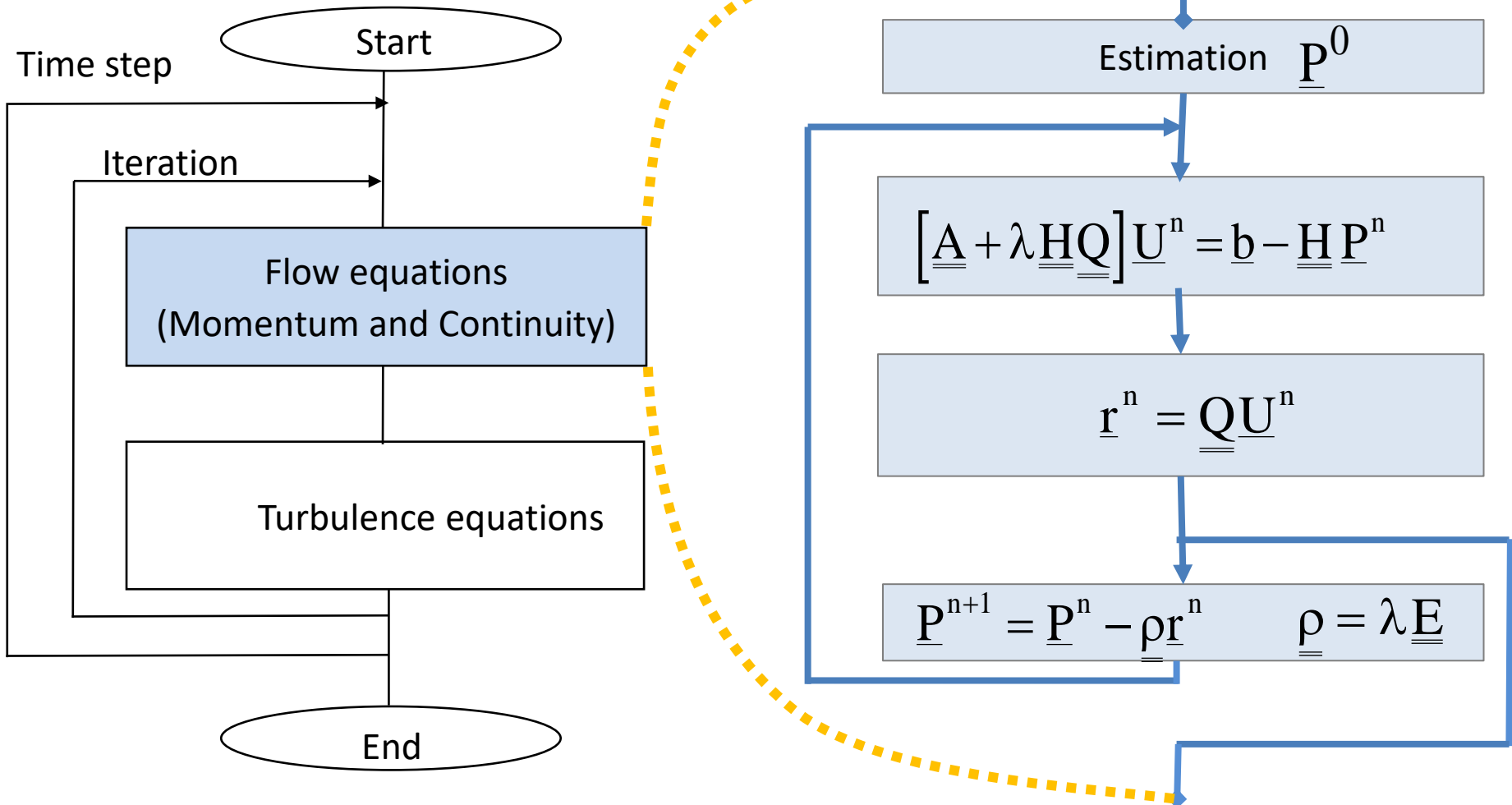
$$\underline{P}^{n+1} = \underline{P}^n - \rho \underline{r}^n$$

Suitable choice of conditioning matrix:

$$\underline{\rho} = \lambda \underline{E}$$

Iteration

Solution process



Pressure correction

Two – stage correction methods

Discretized equations

$$\begin{bmatrix} \underline{\underline{A}} & \underline{\underline{H}} \\ \underline{\underline{Q}} & 0 \end{bmatrix} \begin{bmatrix} \underline{\underline{U}} \\ \underline{\underline{P}} \end{bmatrix} = \begin{bmatrix} \underline{\underline{b}} \\ 0 \end{bmatrix}$$

Momentum equation

Continuity equation

LU – Decomposition (lower and upper triangle matrices)

$$\begin{bmatrix} \underline{\underline{A}} & 0 \\ \underline{\underline{Q}} & -\underline{\underline{Q}}\underline{\underline{A}}^{-1}\underline{\underline{H}} \end{bmatrix} \underbrace{\begin{bmatrix} \underline{\underline{E}} & \underline{\underline{A}}^{-1}\underline{\underline{H}} \\ 0 & \underline{\underline{E}} \end{bmatrix} \begin{bmatrix} \underline{\underline{U}}^{n+1} \\ \underline{\underline{P}}^{n+1} \end{bmatrix}}_{\begin{bmatrix} \underline{\underline{U}}^* \\ \underline{\underline{P}}^* \end{bmatrix}} = \begin{bmatrix} \underline{\underline{b}} \\ 0 \end{bmatrix}$$

Pressure correction

Forward substitution

$$\begin{bmatrix} \underline{\underline{A}} & 0 \\ \underline{\underline{Q}} & -\underline{\underline{Q}}\underline{\underline{A}}^{-1}\underline{\underline{H}} \end{bmatrix} \begin{bmatrix} \underline{U}^* \\ \underline{P}^* \end{bmatrix} = \begin{bmatrix} \underline{b} \\ 0 \end{bmatrix}$$

Backward substitution

$$\begin{bmatrix} \underline{\underline{E}} & \underline{\underline{A}}^{-1}\underline{\underline{H}} \\ 0 & \underline{\underline{E}} \end{bmatrix} \begin{bmatrix} \underline{U}^{n+1} \\ \underline{P}^{n+1} \end{bmatrix} = \begin{bmatrix} \underline{U}^* \\ \underline{P}^* \end{bmatrix}$$

Procedure

$$\underline{\underline{A}}\underline{U}^* = \underline{b}$$

$$\underline{\underline{Q}}\underline{\underline{A}}^{-1}\underline{\underline{H}}\underline{P}^* = \underline{\underline{Q}}\underline{U}^*$$

$$\underline{P}^{n+1} = \underline{P}^*$$

$$\underline{U}^{n+1} = \underline{U}^* - \underline{\underline{A}}^{-1}\underline{\underline{H}}\underline{P}^{n+1}$$

Pressure correction

Calculation of A^{-1} to expensive

=> Approximation of A^{-1}

$$\underline{\underline{A}} \underline{\underline{U}}^* = \underline{\underline{b}}$$



LSE for velocities

$$\underline{\underline{Q}} \underline{\underline{B}}_1 \underline{\underline{H}} \underline{\underline{P}}^* = \underline{\underline{Q}} \underline{\underline{U}}^*$$



Simplified equation for pressure

$$\underline{\underline{P}}^{n+1} = \underline{\underline{P}}^*$$



Pressure update

$$\underline{\underline{U}}^{n+1} = \underline{\underline{U}}^* - \underline{\underline{B}}_2 \underline{\underline{H}} \underline{\underline{P}}^{n+1}$$



Velocity update

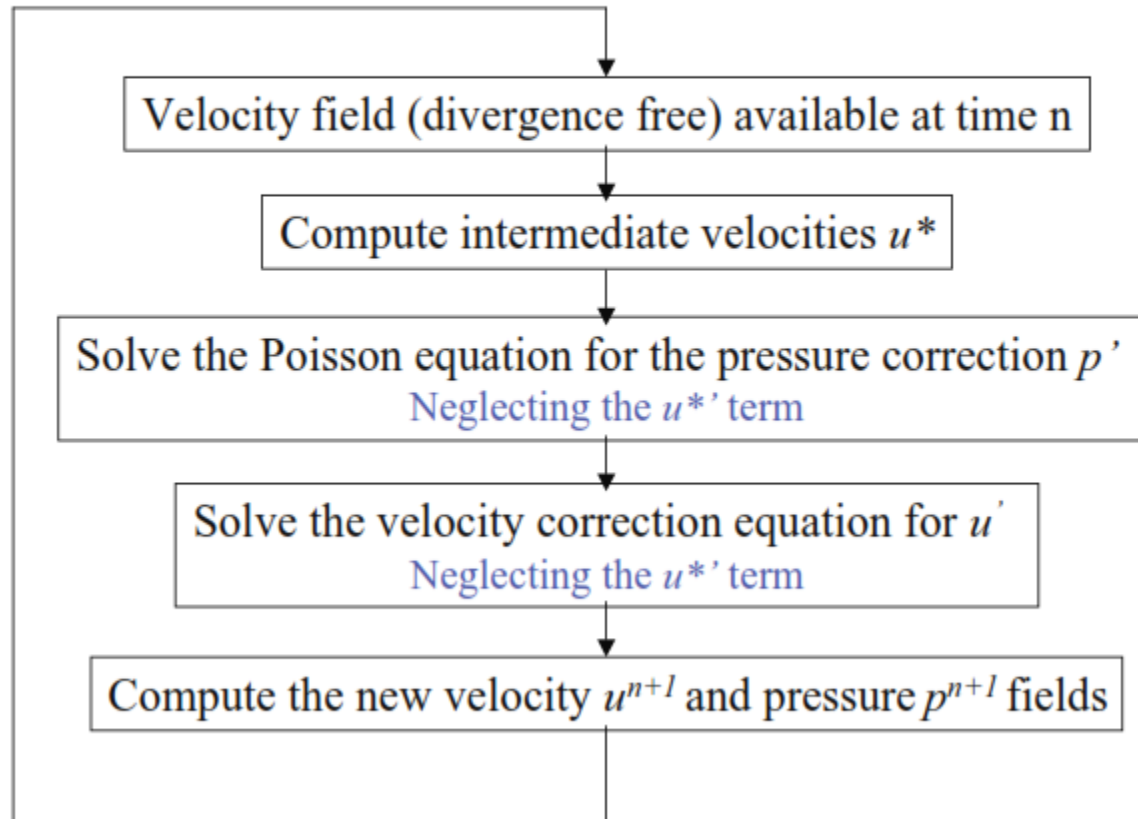
Various pressure correction methods differ by the choice of the approximation for B_1 und B_2

Simplest choice for unsteady flow (projection method)

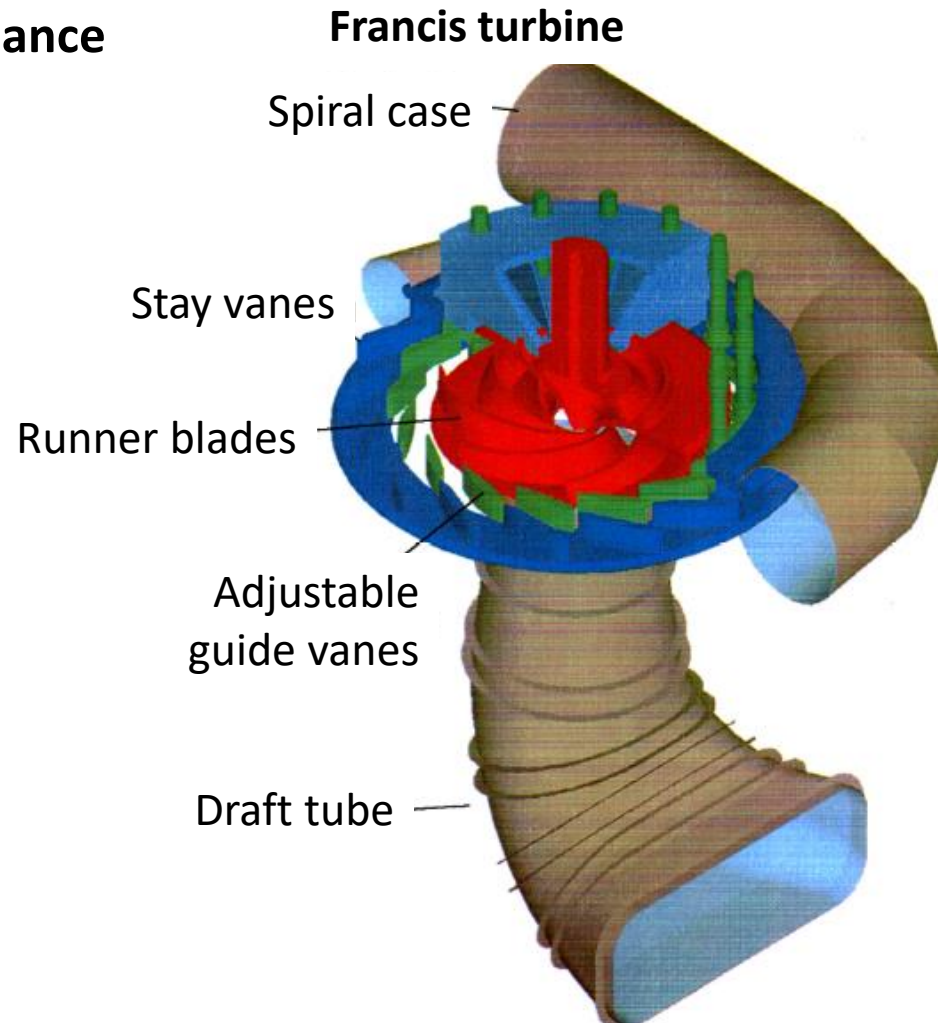
$$\underline{\underline{B}}_1 = \underline{\underline{B}}_2 \approx \Delta t \underline{\underline{E}}$$

Implicit pressure-based scheme for NS equations (SIMPLE)

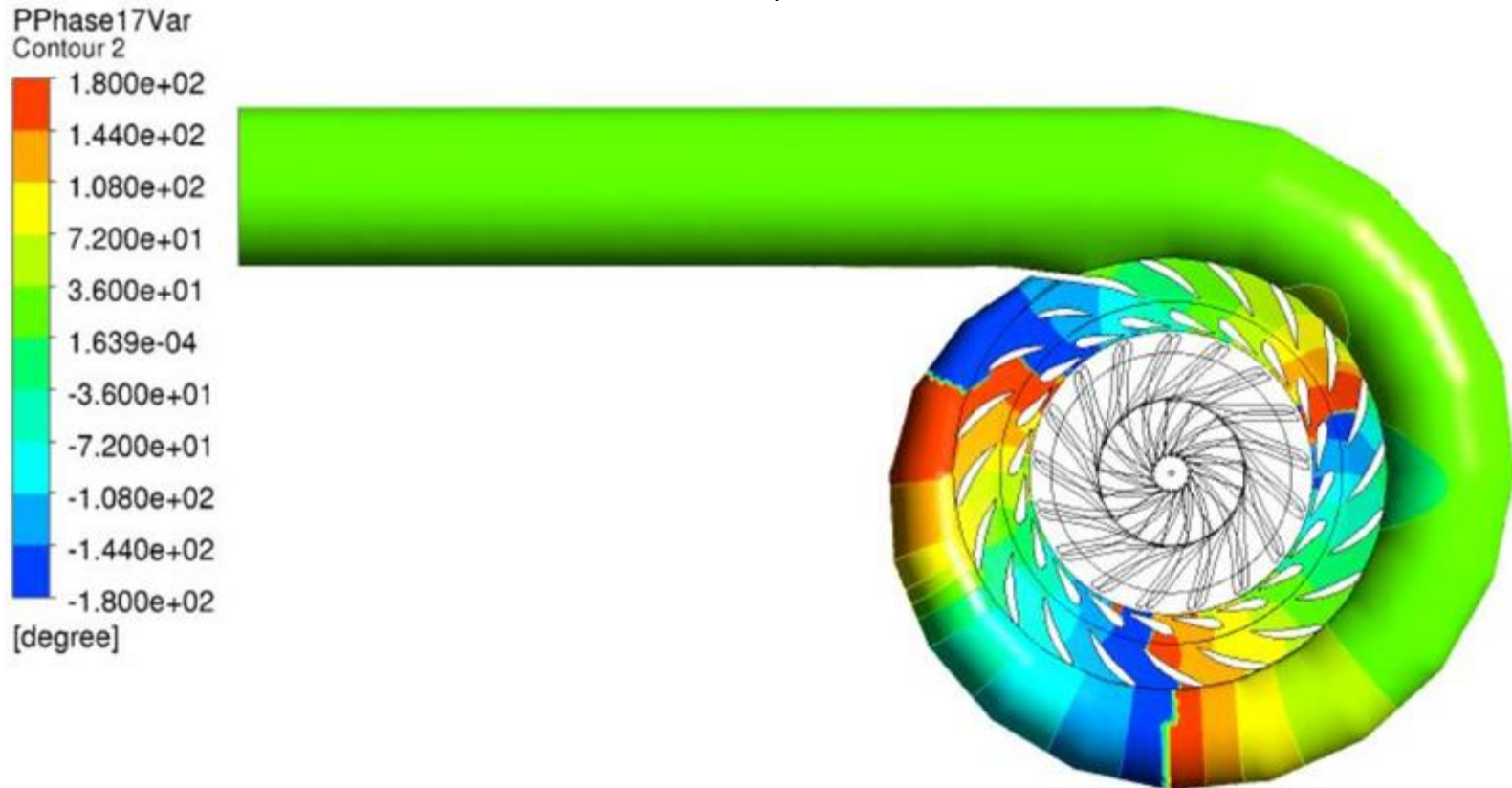
SIMPLE: Semi-Implicit Method for Pressure-Linked Equations



Example: Phase resonance

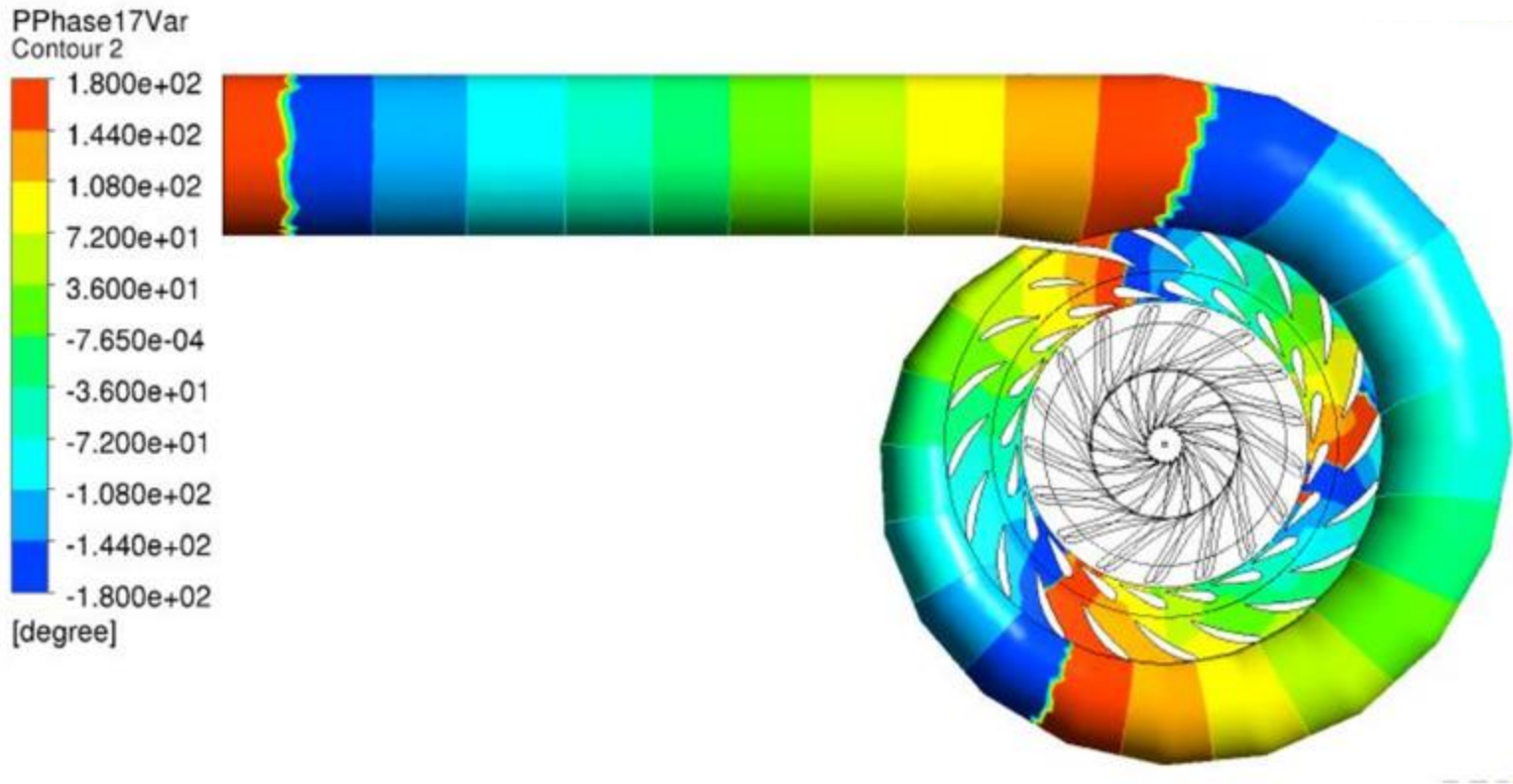


Incompressible calculation



Ruchonnet, N. et al., Simulation of phase resonance in radial hydraulic machines, Vienna Hydro 2014

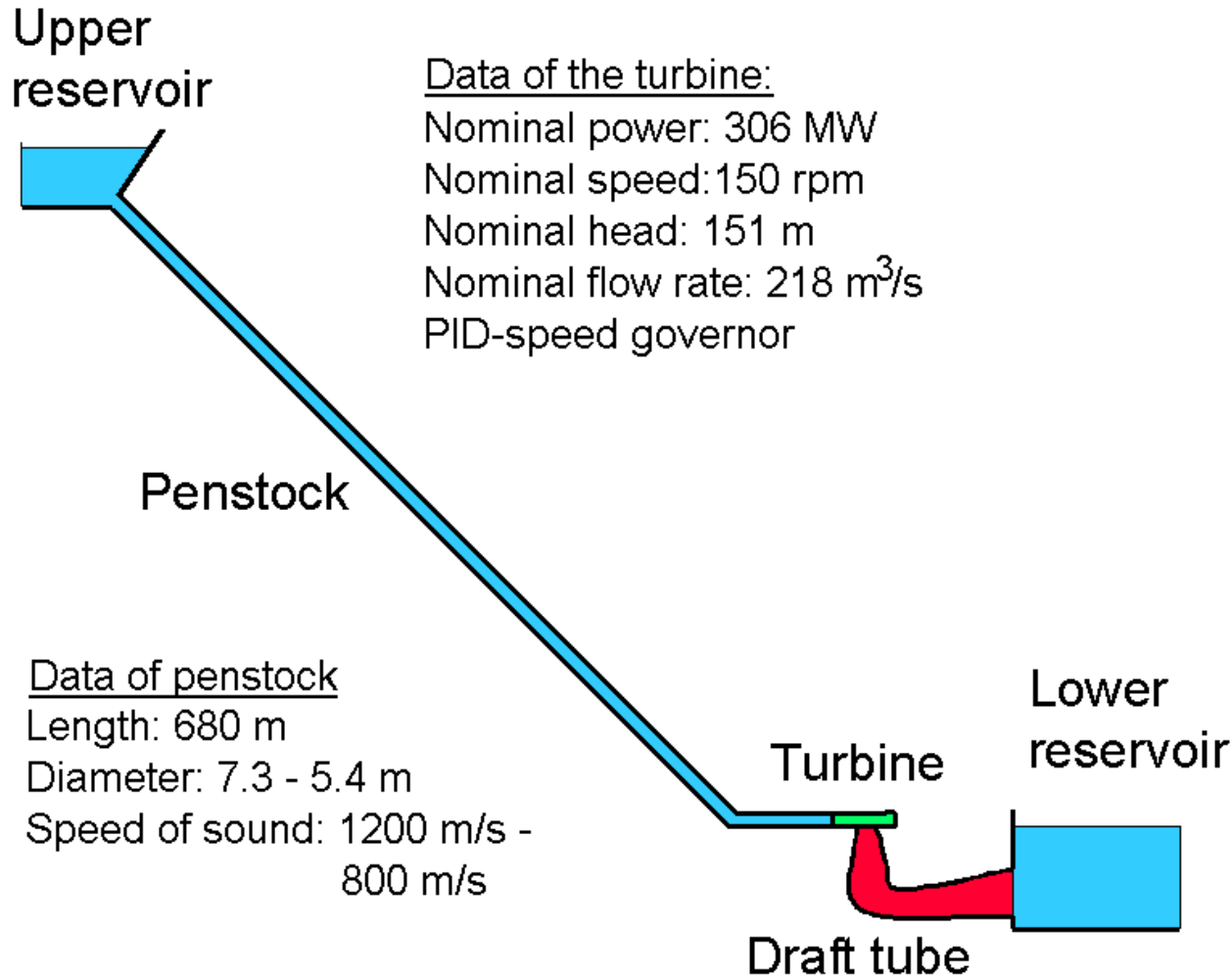
Compressible calculation



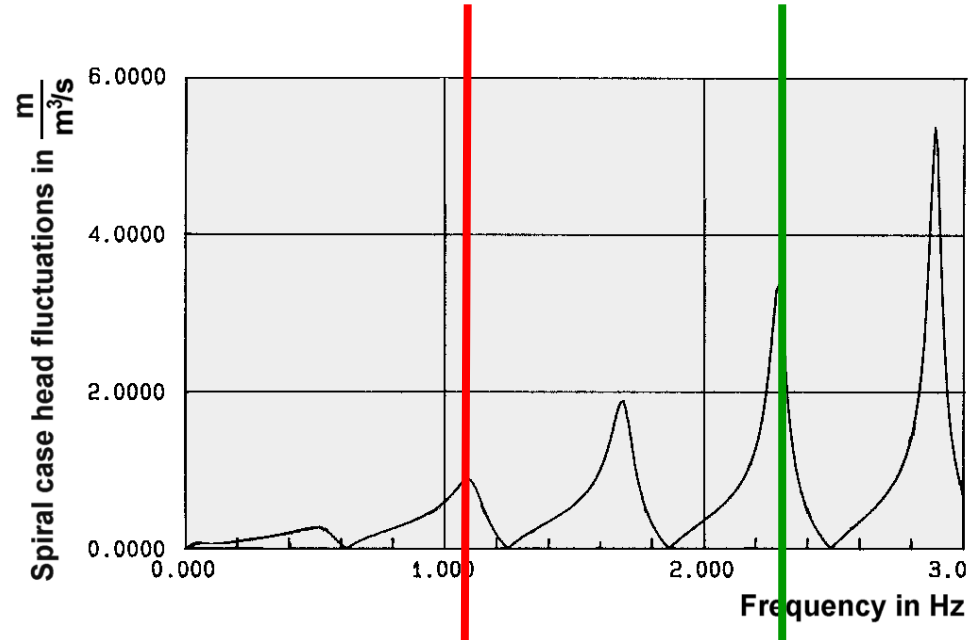
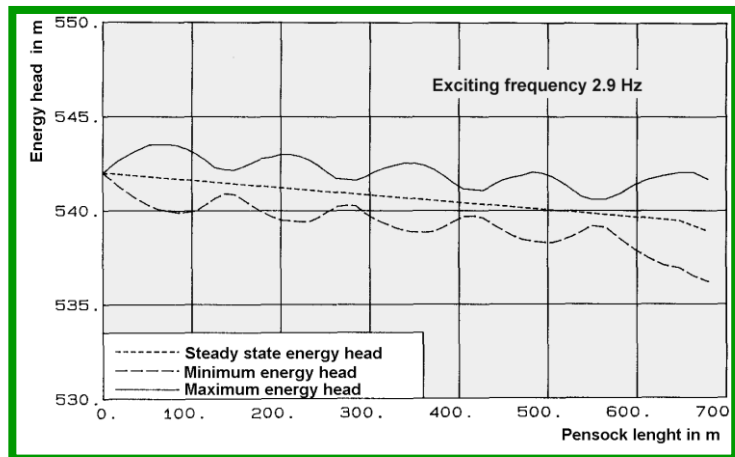
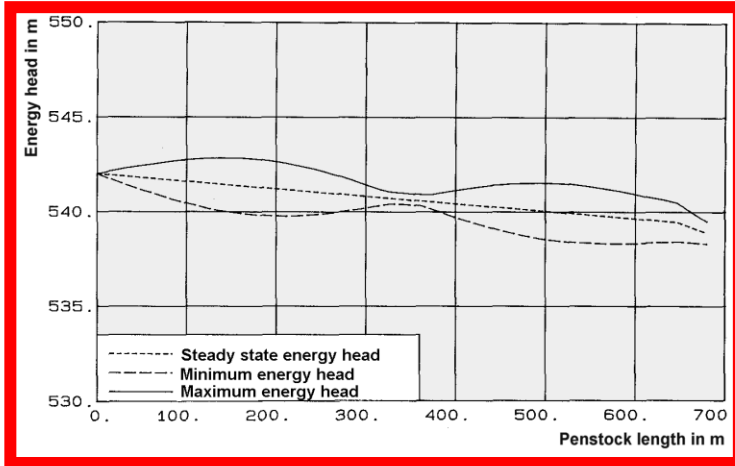
Ruchonnet, N. et al., Simulation of phase resonance in radial hydraulic machines, Vienna Hydro 2014

Coupled compressible – incompressible calculation

Example



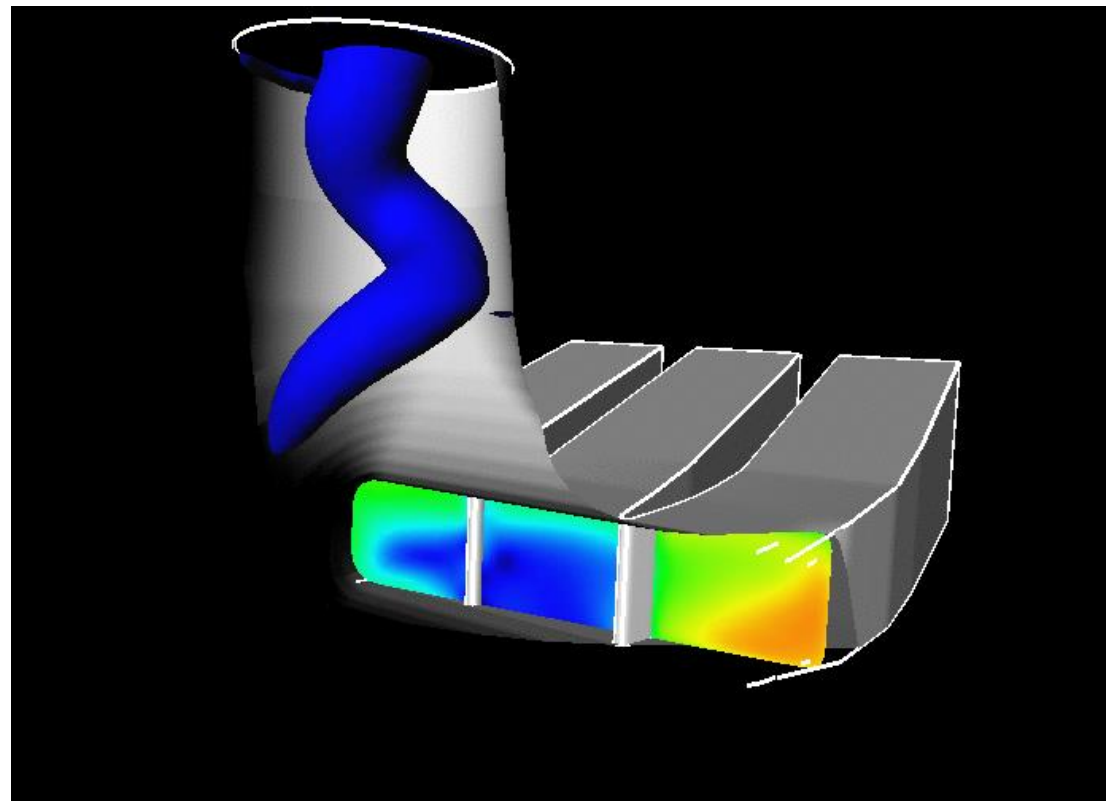
Resonance analysis

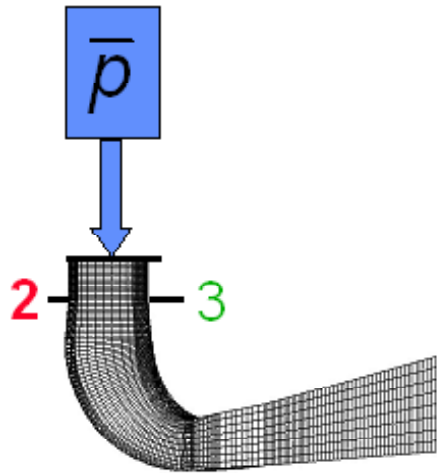


Elbow draft tube

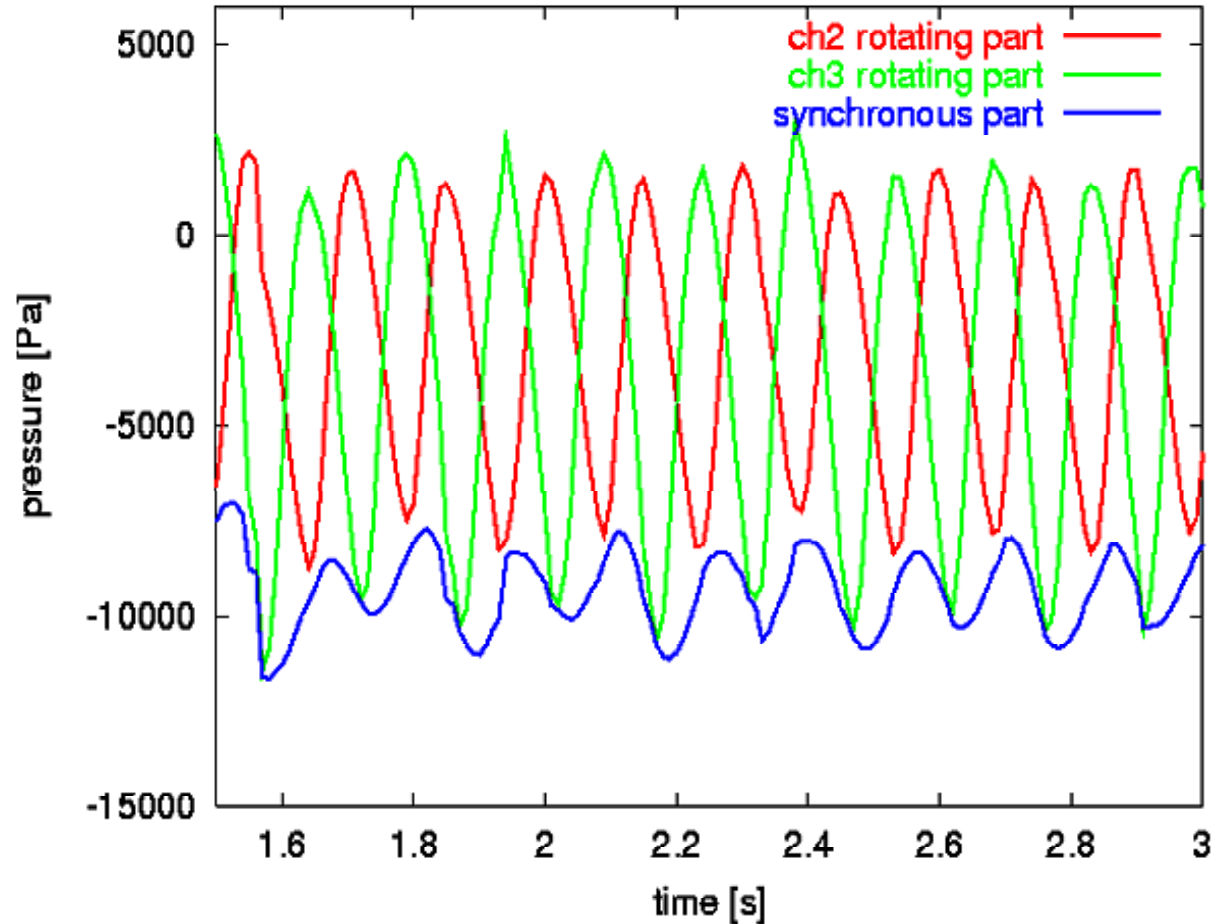
- Unstable flow can result in a vortex rope
- Vortex rope leads to a rotating pressure field
- Dynamic loading on the draft tube structure
- Synchronous pressure pulsation

at the draft tube inlet

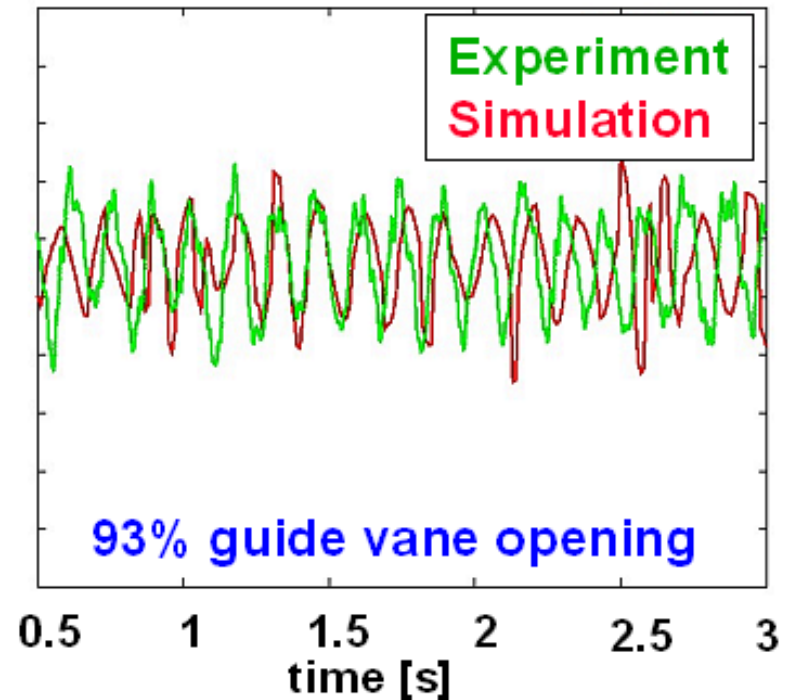
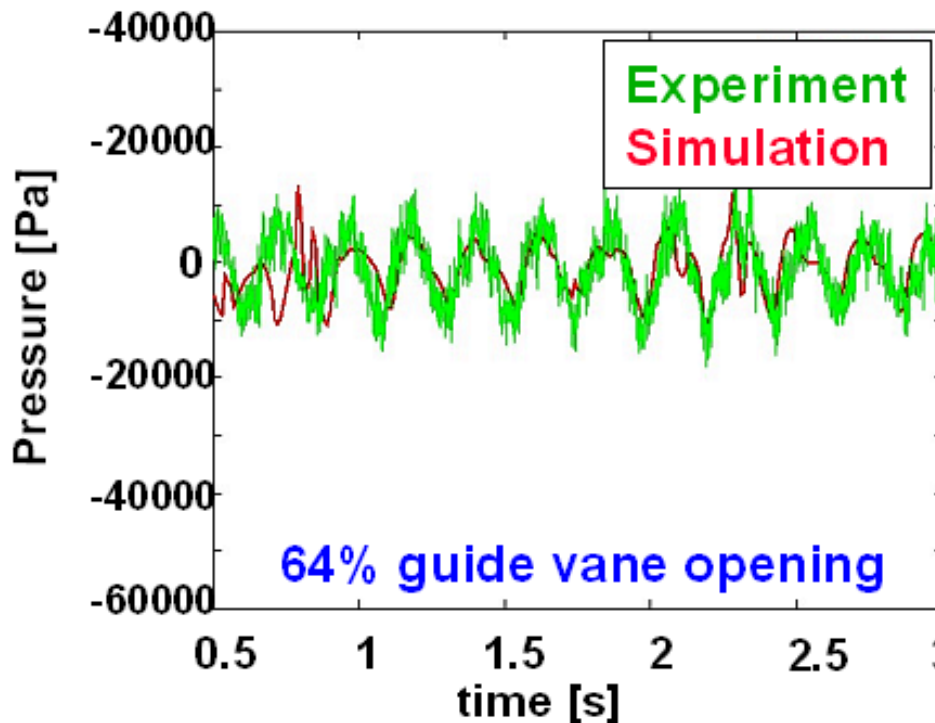
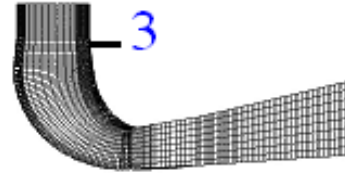




Modellturbine

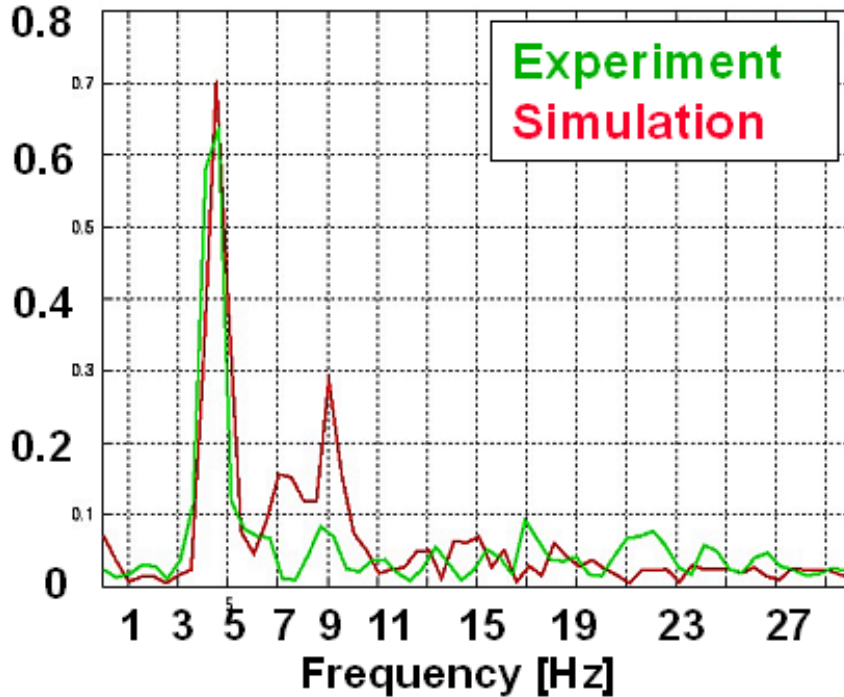


Vortex rope vibration

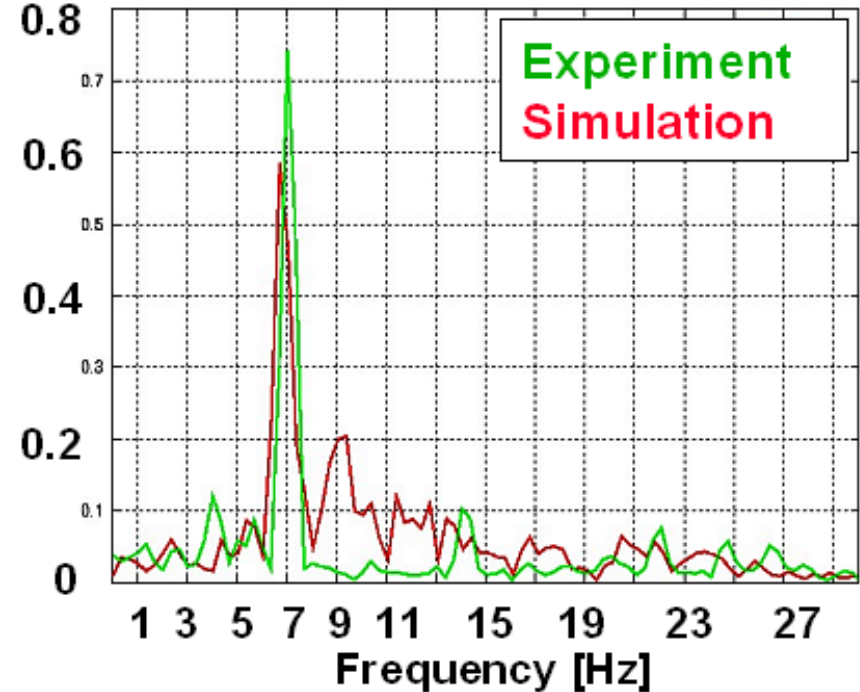


Model turbine

64% Guide vane opening



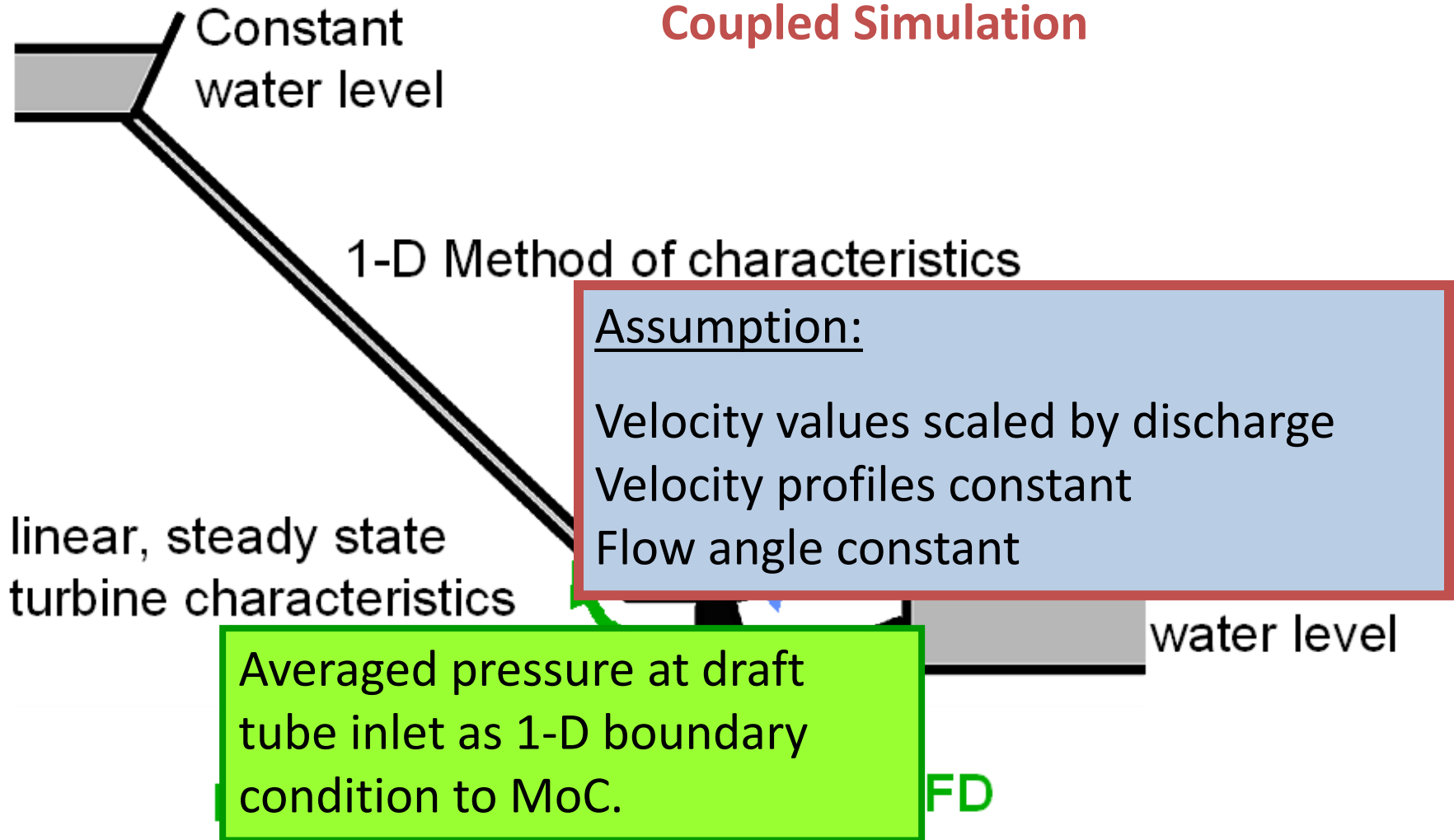
93% Guide vane opening

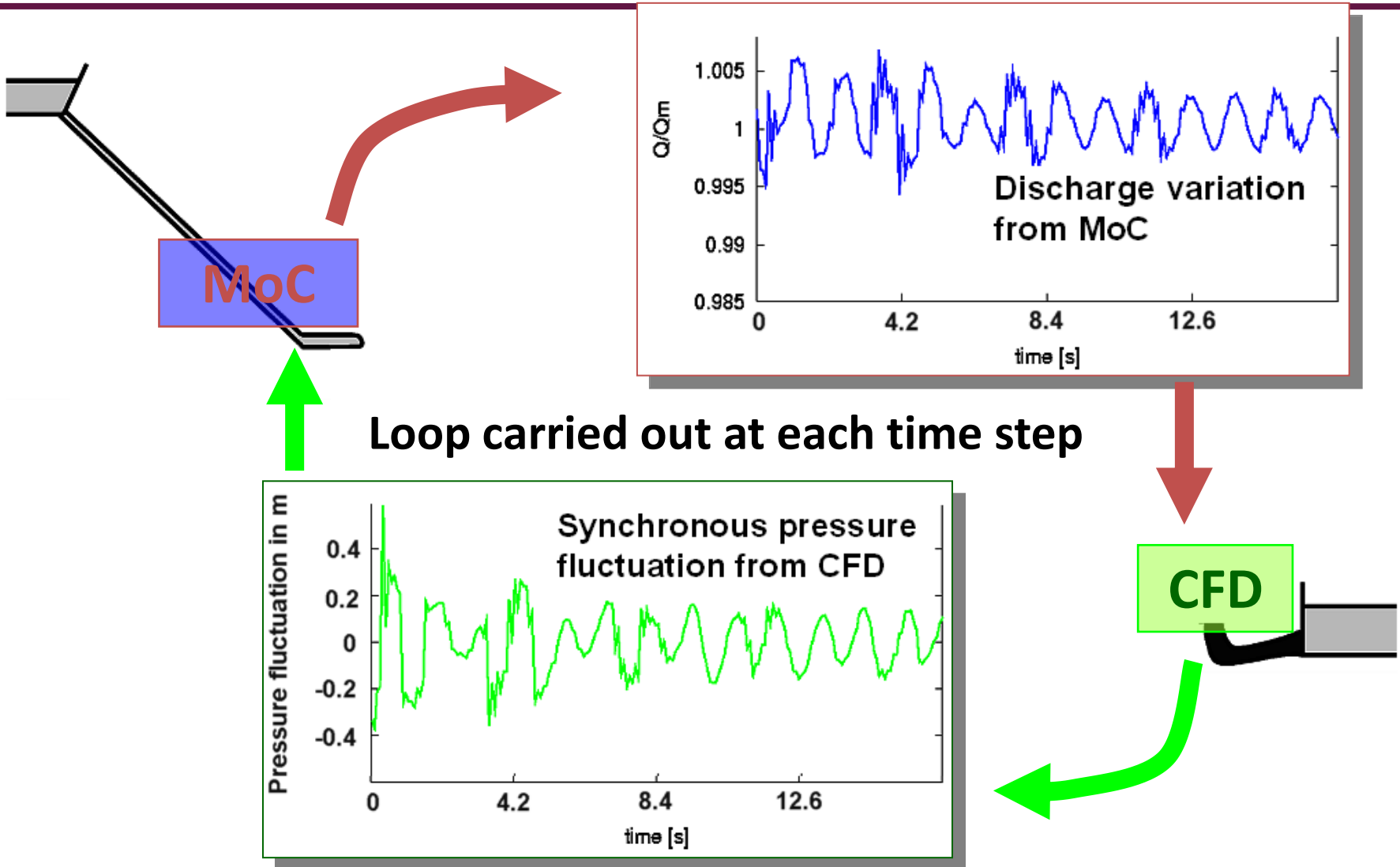


Model turbine

If the excitation is in resonance with the penstock, it can result in extreme pressure amplitudes

Coupled Simulation





Summary

- Relevant physical effects determine if a flow shall be treated incompressible or compressible
 - Applying compressible calculation for low Ma leads to stiff problems, immense computational effort
- Gas flows can be treated incompressible up to a $Ma = 0.3$
- Liquid flows must be treated compressible when pressure waves play an important role (resonance phenomena)
- Physical relevance For incompressible flows there exists no direct equation to determine the pressure. Therefore, a special treatment is required:
 - Artificial compressibility
 - Pressure correction
 - Poisson equation
 - Others
- Coupled simulations (partly compressible, partly incompressible) can be a good compromise